MATHEMATISCHES INSTITUT
DER UNIVERSITÄT MÜNCHEN
Prof. Dr. S. Bachmann / Dr. P. Soneji

WS 2014/15
– Blatt 12 –
14.1.2015

# PDG I
# (Zentralübung)

## Problem Sheet 12

Throughout this problem sheet, let $\Omega$ be an open subset of $\mathbb{R}^n$.

### Question 1

Recall the *Fundamental Lemma of the Calculus of Variations*: if $u \in L^1_{\text{loc}}(\Omega)$ and

$$\int_\Omega u(x)\varphi(x)\,\mathrm{d}x = 0 \quad \text{for all } \varphi \in C_c^\infty(\Omega)$$

then $u = 0$.

(a) Suppose $v_1, v_2 \in L^1_{\text{loc}}(\Omega)$ are both weak derivatives of $u \in L^1_{\text{loc}}(\Omega)$. Show that $v_1 = v_2$ (so weak derivatives are unique).

(b) Suppose

$$\int_\Omega u(x)\varphi(x)\,\mathrm{d}x = 0 \quad \forall \varphi \in C_c^\infty(\Omega) \text{ such that } \int_\Omega \varphi(x)\,\mathrm{d}x = 0\,.$$

Show that $u$ is constant almost everywhere in $\Omega$.

### Question 2

Suppose $f, g \in L^1_{\text{loc}}(\mathbb{R})$ are weakly differentiable. Now consider $u \in L^1_{\text{loc}}(\mathbb{R}^2)$ defined by

$$u(x, y) := f(x) + g(y)\,.$$

Show that $u$ is weakly differentiable with respect to $x$ and $y$, with

$$\frac{\partial u}{\partial x}(x, y) = f'(x)\,, \quad \frac{\partial u}{\partial y}(x, y) = g'(y)\,,$$

where $f'$ and $g'$ are the weak derivatives of $f$ and $g$.

### Question 3

Let $u \in W^{1,p}(\Omega)$, $1 \le p \le \infty$. Let $E$ be an open subset of $\Omega$. Show that $u \in W^{1,p}(E)$, and that the weak derivatives of $u$ on $E$ are the same as the weak derivatives of $u$ on $\Omega$ restricted to $E$.

**Question 4**

Let $B$ be the unit ball in $\mathbb{R}^2$ and fix $0 < s < 1$. Consider the map

$$u(x) = |x|^{-s}.$$

(i) Show that $u \in L^p(B)$ for $1 \le p < \frac{2}{s}$.

(ii) Compute the (strong) partial derivatives $\partial u/\partial x_1$, $\partial u/\partial x_2$ of $u$ on $B \setminus \{(0,0)\}$.

(iii) Now suppose $1 \le p < \frac{2}{s+1}$. Show that each partial derivative $\partial u/\partial x_i$ above satisfies

$$\int_B \left| \frac{\partial u}{\partial x_i} \right|^p \mathrm{d}x < \infty .$$

(iv) Show that the derivatives computed in part (ii) are in fact weak derivatives of $u$ on all of $B$. (Deal with the singularity at $x = 0$ by considering a ball $B_\epsilon$, of small radius $\epsilon > 0$, splitting $B$ into $B_\epsilon$ and $B \setminus B_\epsilon$, applying Gauss-Green, and letting $\epsilon \searrow 0$.)

(iv) Conclude that $u \in W^{1,p}(B)$ for $1 \le p < \frac{2}{s+1}$.

*Hint:* It is helpful to use polar coordinates throughout. For part (iii), find an upper bound for the integrand that depends only on $|x|$.

**Deadline for handing in: 0800 Wednesday 21 January**

*Please put solutions in Box 17, 1st floor (near the library)*

Homepage: `http://www.mathematik.uni-muenchen.de/~soneji/pde1.php`