PDG I
(Zentralübung)

Problem Sheet 1

Question 1

(a) Prove the *Multinomial Theorem*:

$$(x_1 + \ldots + x_n)^k = \sum_{|\alpha|=k} \binom{|\alpha|}{\alpha} x^\alpha \,,$$

where

$$\binom{|\alpha|}{\alpha} := \frac{|\alpha|!}{\alpha!} \,, \quad \alpha! := \alpha_1!\alpha_2!\ldots\alpha_n!$$

and $x^\alpha = x_1^{\alpha_1} \ldots x_n^{\alpha_n}$.

The sum is taken over all multi-indices $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$ with $|\alpha| = k$.

(b) Prove *Leibniz's formula*:

$$D^\alpha(uv) = \sum_{\beta\le\alpha} \binom{\alpha}{\beta} D^\beta u \, D^{\alpha-\beta} v \,,$$

where u, $v \colon \mathbb{R}^n \to \mathbb{R}$ are smooth,

$$\binom{\alpha}{\beta} := \frac{\alpha!}{\beta!(\alpha-\beta)!} \,,$$

and $\beta \le \alpha$ means $\beta_i \le \alpha_i$ for $i = 1, \ldots, n$.

Question 2

Classify each of the following partial differential equations as *linear*, *semilinear*, *quasilinear* or *fully nonlinear*. Also determine the order of each equation. In each case we have $u \colon \Omega \to \mathbb{R}$ for some open subset Ω of $\mathbb{R}^n$.

(a) $u_{x_1x_2} + u_{x_2x_3} = 0$

(b) $u|u_{x_1}|^2 + u_{x_1x_2} = 0$

(c) $u|u_{x_1}|^2 u_{x_1x_2} = 0$

(d) $x_1x_2^2\, u_{x_1x_2} = x_2 \sin(x_1)$

(e) $uu_{x_1} + |u_{x_1 x_2}|^2 = 0$

(f) $-\sum_{i=1}^{n} (b^i u)_{x_i} = 0$, where $b = (b^1, \dots b^n) \in \mathbb{R}^n$

(g) $-\Delta u = f(u)$ (recall $\Delta u := \sum_{i=1}^{n} u_{x_i x_i}$)

(h) $iu_t + \Delta u = f(|u|^2)u$ (here $u \colon (0,T) \times \Omega \to \mathbb{R}$)

(i) $\det(D^2 u) = f$

(j) $\operatorname{div}(|\nabla u|^{p-2} \nabla u) = 0.$

Question 3

Show that functions of the form $u(x,y) = f(x) + g(y)$, where f and g both belong to $C^1(\mathbb{R})$, are solutions to the partial differential equation

$$u_{xy}(x,y) = 0 \quad \text{on } \mathbb{R}^2 .$$

What is the order of this PDE? Is such a solution u necessarily in $C^2(\mathbb{R}^2)$?

Question 4

Write down an explicit formula for a function u solving the initial value problem

$$\begin{cases} u_t + b \cdot Du + cu = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ u = g & \text{on } \mathbb{R}^n \times \{t = 0\} . \end{cases}$$

Here, $g \colon \mathbb{R}^n \to \mathbb{R}$ is smooth, and $c \in \mathbb{R}$, $b \in \mathbb{R}^n$ are constants.

Hint: Use the method of characteristics. Here, recognize the left hand side of the equation as the derivative of a product of u with a simple function.

Deadline for handing in: 0800 Wednesday 22 October

Please put solutions in Box 17, 1st floor (near the library)