MATHEMATISCHES INSTITUT
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SS 2015
– Blatt 8 –
10.6.2015

PDG II
(Tutorium)

Tutorial 8

In the following, U always denotes an open, bounded subset of $\mathbb{R}^n$. Let $f\colon \mathbb{R}^N \to \mathbb{R}$ be a continuous function and define, for maps $u\colon U \to \mathbb{R}^N$, the variational integral

$$F(u, U) := \int_U f(u(x))\,\mathrm{d}x\,.$$

Exercise 1

Suppose $f \geq 0$ and $1 \leq p \leq \infty$. Suppose (u_j), $u \subset L^p(U;\mathbb{R}^N)$, and $u_j \to u$ strongly in $L^p(U;\mathbb{R}^N)$. Prove that

$$\liminf_{j\to\infty} F(u_j, U) \geq F(u, U)\,.$$

(i.e. F is *strongly lower semicontinuous* in $L^p(U;\mathbb{R}^N)$.)

Exercise 2

Suppose (u_j), $u \subset L^\infty(U;\mathbb{R}^N)$, and $u_j \to u$ strongly in $L^\infty(U;\mathbb{R}^N)$. Prove that

$$F(u_j, U) \to F(u, U)\,.$$

In fact, with these assumptions, we even have $(f \circ u_j) \to (f \circ u)$ strongly in $L^\infty(\Omega)$. Can you show this too?

Exercise 3

Now assume that f satisfies the growth condition

$$|f(v)| \leq C(1 + |v|^p) \quad \text{for all } v \in \mathbb{R}^N$$

for some exponent $1 \leq p < \infty$ and a fixed constant $C > 0$. Suppose (u_j), $u \subset L^p(U;\mathbb{R}^N)$, and $u_j \to u$ strongly in $L^p(U;\mathbb{R}^N)$. Prove that $F(u_j, U) \to F(u, U)$.