MATHEMATISCHES INSTITUT
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PDG II
(Tutorium)

Tutorial 7

In the following, U always denotes an open, bounded subset of $\mathbb{R}^n$ with smooth boundary.

Exercise 1

This exercise concerns the proof of Theorem 2.4 (”Energy estimate”) from the lectures (see lectures for notation.)

(a) Give an explicit expression for α.

(b) With the choice of ε given in the lectures, find an explicit expression for C, β and γ.

(c) Can one improve upon the constant γ by making a different choice for ε?

(d) Which expression for γ does one get if one does *not* use Poincaré's inequality?

(e) Let

$$Lu = -\sum_{i,j=1}^{n} (a^{ij} u_{x_i})_{x_j} + cu$$

be uniformly elliptic. Prove that there exists a constant $\mu > 0$ such that the corresponding bilinear form $B[\ ,\ ]$ satisfies the hypotheses of the Lax-Milgram Theorem, provided

$$c(x) \geq -\mu, \quad (x \in U).$$

Note: ”Explicit expression” in this context means a function of a^{ij}, b^i, c, θ, and possibly other data.

Exercise 2

This exercise concerns the proof of Theorem 2.13 in the lectures (for notation, see the lectures): Prove the claim that $v - K^*v = 0$ if and only if v is a weak solution of

$$\begin{cases} L^*v = 0 & \text{in } U, \\ v = 0 & \text{on } \partial U. \end{cases}$$

Exercise 3

Let $A : H \to H$ be a bounded operator on a Hilbert space H, and assume that there exists $\beta > 0$ such that

$$\|Au\| \geq \beta\|u\|, \quad u \in H.$$

Prove that $\mathrm{Ran}(A)$ is closed.

Exercise 4

A function $u \in H_0^2(U)$ is a weak solution to the boundary-value problem for the *biharmonic equation*

$$\begin{cases} \Delta^2 u = f & \text{in } U, \\ u = \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial U, \end{cases} \tag{1}$$

provided

$$\int_U \Delta u \Delta v \, dx = \int_U f v \, dx$$

for all $v \in H_0^2(U)$. Given $f \in L^2(U)$ prove that there exists a unique weak solution of (1).