PDG II
(Tutorium)

Tutorial 5

Exercise 1

Fill in the details in the proof of Theorem 1.19 in the Lecture. More precisely, let $\{u_m\}_{m=1}^{\infty} \subset C_c^{\infty}(\mathbb{R}^n)$ such that $u_m \to \overline{u}$ in $W^{1,p}(\mathbb{R}^n)$. Use the Gagliardo-Nirenberg-Sobolev inequality to conclude that $u_m \to \overline{u}$ in $L^{p^*}(\mathbb{R}^n)$ (see the Lecture for notation) and

$$\|\overline{u}\|_{L^{p^*}(\mathbb{R}^n)} \leq C\|D\overline{u}\|_{L^p(\mathbb{R}^n)}.$$

Moreover, conclude that the Gagliardo-Nirenberg-Sobolev inequality (Theorem 1.16 in the Lecture) holds for $u \in W^{1,p}(\mathbb{R}^n)$, not just for $u \in C_c^{\infty}(\mathbb{R}^n)$.

Exercise 2

Let $U \subset \mathbb{R}^n$ be open and bounded and U' open, with $U \subset U'$. For $u \in H_0^1(U)$ define its extension $\widetilde{u}$ to U' by

$$\widetilde{u}(x) := \begin{cases} u(x) & x \in U, \\ 0 & x \in U' \setminus U. \end{cases}$$

Prove that $\widetilde{u} \in H_0^1(U')$ and that

$$\|\widetilde{u}\|_{H_0^1(U')} = \|u\|_{H_0^1(U)}.$$

Exercise 3

Prove Morrey's inequality (Theorem 1.21 in the Lecture).