MATHEMATISCHES INSTITUT
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PDG II
(Tutorium)

Tutorial 3

In the following, $U \subset \mathbb{R}^n$ will always denote an open set.

Exercise 1

Prove in detail that:

(a) $(D^\alpha u_j) \subset L^p(U)$ is a Cauchy sequence in $L^p(U)$ for all $|\alpha| \leq k$ if and only if (u_j) is a Cauchy sequence in $W^{k,p}(U)$.

(b) $D^\alpha u_j \to D^\alpha u$ in $L^p(U)$ for all $|\alpha| \leq k$ if and only if $u_j \to u$ in $W^{k,p}(U)$.

Exercise 2

Assume U is bounded. Do some of the missing details in the proof of Theorem 1.10 in the Lecture:

(a) Let $U_i = \{x \in U : \text{dist}(x, \partial U) > 1/i\}$ ($i \in \mathbb{N}$). Prove that U_i is open, and that $U = \cup_{i=1}^\infty U_i$.

(b) Prove that $V_i := U_{i+3} \setminus \overline{U_{i+1}}$ is open, and that there exists V_0 open, with $V_0 \subset\subset U$, such that $U = \cup_{i=0}^\infty V_i$.

Exercise 3

Assume U is bounded. A particular argument was used several times in the Lecture. Prove it (for notation, see the Lecture):

(a) In the proof of Theorem 1.11: the inequality ($v = \sum_{i=0}^N \zeta_i v_i$, $|\alpha| \leq k$)

$$\|D^\alpha v - D^\alpha u\|_{L^p(U)} \leq C \sum_{i=0}^N \|v_i - u\|_{W^{k,p}(V_i)} \quad (\leq C(N+1)\delta\,).$$

(b) In the proof of Theorem 1.12 ("local to global"): the inequality

$$\|\overline{u}\|_{W^{1,p}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(U)} \quad (\overline{u} = \sum_{i=0}^N \zeta_i \overline{u}_i\,).$$

Exercise 4

Assume U is bounded. Do some of the missing details in the proof of Theorem 1.12 in the Lecture (for notation, see the Lecture):

(a) For the case of ∂U flat, $u \in C^1(\overline{B}^+)$, $\overline{u}$ its C^1-extension to B: the inequality

$$\|\overline{u}\|_{W^{1,p}(B)} \leq C\|u\|_{W^{1,p}(B^+)} .$$

(b) "Straightening of the boundary": prove that $\Phi = \Psi^{-1}$ and $\det D\Phi = \det D\Psi = 1$.

(c) After having straightened the boundary, and defined the extension $\overline{u}$ of u via Ψ and Φ: The inequality

$$\|\overline{u}\|_{W^{1,p}(W)} \leq C\|u\|_{W^{1,p}(U)} .$$

Exercise 5

Assume U is bounded. Do some of the missing details in the proof of Theorem 1.12 in the Lecture (for notation, see the Lecture):

After having defined Eu for all $u \in W^{1,p}(U)$ via approximation, prove that:

(a) Eu is independent of the choice of the approximating sequence.

(b) $E\colon W^{1,p}(U) \to W^{1,p}(\mathbb{R}^n)$ is linear.

(c) E is bounded (linear operator) on $W^{1,p}(U)$:

$$\exists C > 0 : \forall u \in W^{1,p}(U) \ : \ \|Eu\|_{W^{1,p}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(U)} .$$