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SS 2015
– Blatt 2 –
29.4.2015

PDG II
(Tutorium)

Tutorial 2

In the following, $U \subset \mathbb{R}^n$ will always denote an open set.

Exercise 1

(a) Let $k \in \{0, 1, \dots\}$, $0 < \gamma \leq 1$. Prove $C^{k,\gamma}(\overline{U})$ is a Banach Space.

(b) Assume that U has a smooth boundary ∂U, and let V be an open set such that $V \subset\subset U$. Show that there exists a smooth function ζ on U such that $\zeta \equiv 1$ on V and $\zeta \equiv 0$ near ∂U. (*Hint:* Take $V \subset W \subset U$ and mollify χ_W.)

Exercise 2

Prove the following inequalities:

(a) Assume u, $v \in L^p(U)$, with $1 < p < \infty$ and $q = p/(p-1)$. Then, for $1 < p \leq 2$,

$$\left\| \frac{u+v}{2} \right\|_p^q + \left\| \frac{u-v}{2} \right\|_p^q \leq \left(\tfrac{1}{2}\|u\|_p^p + \tfrac{1}{2}\|v\|_p^p \right)^{q-1},$$

and for $2 \leq p < \infty$,

$$\left\| \frac{u+v}{2} \right\|_p^p + \left\| \frac{u-v}{2} \right\|_p^p \leq \tfrac{1}{2}\|u\|_p^p + \tfrac{1}{2}\|v\|_p^p .$$

(b) Let $U \subset \mathbb{R}^n$, $V \subset \mathbb{R}^m$ be open sets and $\|u(\cdot, y)\|_{L^p(U)} \in L^1(V)$ for some $1 \leq p < \infty$. Then the map $x \mapsto \int_V u(x,y)\,\mathrm{d}y$ is in $L^p(U)$, and

$$\left(\int_U \left| \int_V u(x,y)\,\mathrm{d}y \right|^p \mathrm{d}x \right)^{1/p} \leq \int_V \|u(\cdot, y)\|_{L^p(U)}\,\mathrm{d}y .$$

Exercise 3

Let $U = (-1, 1)^2 \subset \mathbb{R}^2$. Define

$$u(x) = \begin{cases} 1 - x_1 & \text{if } x_1 > 0,\ |x_2| < x_1 , \\ 1 + x_1 & \text{if } x_1 < 0,\ |x_2| < -x_1 , \\ 1 - x_2 & \text{if } x_2 > 0,\ |x_1| < x_2 , \\ 1 + x_2 & \text{if } x_2 < 0,\ |x_1| < -x_2 . \end{cases}$$

For which $1 \leq p \leq \infty$ does u belong to $W^{1,p}(U)$?