PDG II
(Tutorium)

Tutorial 11

Exercise 1

This exercise concerns the proof of Theorem 2.30 from the lecture (see lecture for notation.) Recall that $v = -D_k^{-h}(\zeta^2 D_k^h u)$. Prove that $v = 0$ on ∂U in the trace sense (hence $v \in H_0^1(U)$).

Exercise 2

This exercise concerns the proof of Theorem 2.30 (see also Theorem 2.26; for notation, see the lecture). Let

$$U = B(0,1) \cap \mathbb{R}^n_+,$$
$$V = B(0,1/2) \cap \mathbb{R}^n_+,$$

and let $u \in W^{1,p}(U)$.

(a) Prove that, for all $i = 1, \dots n-1$ (i.e. $i \neq n$):

$$\int_V |D_i^h u|^p \, dx \leq \int_U |u_{x_i}|^p \, dx \quad \text{(for } |h| \text{ small enough).}$$

(b) Prove that if $u \in L^p(V)$ and if, for some $i \in \{1, \dots, n-1\}$, there exists $C > 0$ such that $\|D_i^h u\|_{L^p(V)} \leq C$ for all $0 < |h| < 1/2$, then the weak derivative u_{x_i} exists, $u_{x_i} \in L^p(V)$, and $\|u_{x_i}\|_{L^p(V)} \leq C$.

Exercise 3

This exercise concerns the proof of Theorem 2.30 from the lecture (see lecture for notation.) Recall that $\tilde{u}(y) = u(\Psi(y))$.

(a) Prove that $\tilde{u} \in H^1(\tilde{U})$.

(b) Prove that $\tilde{u} = 0$ on $\partial\tilde{U} \cap \{y_n = 0\}$ in the trace sense.

(c) Prove that $\tilde{u}$ is a weak solution of $\tilde{L}\tilde{u} = \tilde{f}$ in $\tilde{U}$, with $\tilde{f}(y) = f(\Psi(y))$ and

$$\tilde{L}\tilde{u} = -\sum_{l,k=1}^{n} \left(\tilde{a}^{kl}\tilde{u}_{y_k}\right)_{y_l} + \sum_{k=1}^{n} \tilde{b}^k \tilde{u}_{y_k} + \tilde{c}\tilde{u},$$

and with $\tilde{a}^{kl}$, $\tilde{b}^k$, $\tilde{c}$ as in the lecture.

(d) Prove that $\tilde{a}^{kl} \in C^1(\tilde{U})$.

(e) Prove that $\tilde{L}$ is uniformly elliptic in $\tilde{U}$.

Exercise 4

This exercise concerns the proof of Theorem 2.32 from the lecture (see lecture for notation.)

(a) Prove that $\widetilde{u} = 0$ in the trace sense along $\{x_n = 0\}$.

(b) Prove that $\widetilde{u}$ is a weak solution of $\tilde{L}\widetilde{u} = \widetilde{f}$ in U.

(c) Show that

$$D^{\gamma} L u = a^{nn} D^{\beta} u + R$$

where R is a sum of terms involving at most j derivatives of u with respect to x_n and at most $k+3$ derivatives in total.