MATHEMATISCHES INSTITUT
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PDG II
(Tutorium)

Tutorial 10

Exercise 1

This exercise concerns the proof of Theorem 2.27 in the lecture (for notation, see the lecture).

(a) Prove in detail the bound

$$|A_2| \leq C \int_U \left\{ \zeta |D_k^h Du| \, |D_k^h u| + \zeta |D_k^h Du| \, |Du| + \zeta |D_k^h u| \, |Du| \right\} \, dx$$

and conclude that

$$|A_2| \leq \varepsilon \int_U \zeta^2 |D_k^h Du|^2 \, dx + \frac{C}{\varepsilon} \int_W \left\{ |D_k^h u|^2 + |Du|^2 \right\} \, dx.$$

(b) Prove the bound

$$\int_U |v|^2 \, dx \leq C \int_U \left\{ |Du|^2 + \zeta^2 |D_k^h Du|^2 \right\} \, dx$$

and conclude that

$$|B| \leq \varepsilon \int_U \zeta^2 |D_k^h Du|^2 \, dx + \frac{C}{\varepsilon} \int_U \left\{ f^2 + u^2 + |Du|^2 \right\} \, dx.$$

(c) Finally, prove the bound

$$\int_U \zeta^2 |D_k^h Du|^2 \, dx \leq C \int_U \left\{ f^2 + u^2 + |Du|^2 \right\} \, dx.$$

Exercise 2

This exercise concerns the last part of the proof of Theorem 2.27 (for notation, see the lecture). With $v := \chi^2 u$ and $\sum_{i,j=1}^n \int_U a^{ij} u_{x_i} v_{x_j} \, dx = \int_U \widetilde{f} v \, dx$, prove that

$$\int_U \chi^2 |Du|^2 \, dx \leq C \int_U \left(f^2 + u^2 \right) \, dx.$$

Conclude that

$$\|u\|_{H^1(W)} \leq C \left(\|f\|_{L^2(U} + \|u\|_{L^2(U)} \right).$$

Exercise 3

This exercise concerns the proof of Theorem 2.27 (for notation, see the lecture).

(a) Prove that the conclusion of the theorem still holds if one only assumes $b^i, c \in L^\infty_{\text{loc}}(U)$.

(b) Does it hold if one only assumes $a^{ij} \in W^{1,\infty}(U)$ (and still uniform ellipticity)?

Exercise 4

This exercise concerns the proof of Theorem 2.28 from the lecture (see lecture for notation.)

(a) Recall that u is a weak solution of $Lu = f$ in U, $V \subset\subset W \subset\subset U$ and $v = (-1)^{|\alpha|} D^\alpha \widetilde{v}$ for $\widetilde{v} \in C_c^\infty(W)$. Prove that $\widetilde{u}$ satisfies

$$B[\widetilde{u}, \widetilde{v}] = (\widetilde{f}, \widetilde{v})_{L^2(U)},$$

with

$$\widetilde{f} := D^\alpha f - \sum_{\substack{\beta \le \alpha \\ \beta \neq \alpha}} \binom{\alpha}{\beta} \left\{ - \sum_{i,j=1}^n \left(D^{\alpha-\beta} a^{ij} D^\beta u_{x_i} \right)_{x_j} \right. \\ \left. + \sum_{i=1}^n D^{\alpha-\beta} b^i D^\beta u_{x_i} + D^{\alpha-\beta} c D^\beta u \right\}$$

(b) Conclude that $\widetilde{u}$ is a weak solution of $\widetilde{L}\widetilde{u} = \widetilde{f}$ in W.

Exercise 5

This exercise concerns the proof of Theorem 2.28 (for notation, see the lecture). Recall that the coefficients $a^{ij}, b^i, c \in C^{m+1}(U)$ $(i, j = 1, \dots, n)$. Prove that this is indeed *enough* for the conclusion of the theorem to hold. (i.e. $b^i, c \in L^\infty(U)$, for example, is not needed.)