MATHEMATISCHES INSTITUT
DER UNIVERSITÄT MÜNCHEN
Prof. Dr. T. Sørensen / Dr. P. Soneji

SS 2015
– Blatt 1 –
22.4.2015

PDG II
(Tutorium)

Tutorial 1

Exercise 1

We shall first recall some key facts and theorems from measure theory and integration, especially the convergence theorems:

- The Monotone Convergence Theorem
- Fatou's Lemma
- The Dominated Convergence Theorem

Exercise 2

Let $(X, \mathscr{M}, \mu)$ be a measure space with $\mu(X) < \infty$. Let f, $f_j \colon X \to [-1, 1]$ be a sequence of measurable functions. Consider the statements

(a) $\forall \epsilon > 0$, $\mu(\{x : |f(x) - f_j(x)| > \epsilon\}) \to 0$ as $j \to \infty$ (convergence in measure)

(b) $\int_X |f(x) - f_j(x)| \, \mathrm{d}\mu(x) \to 0$ as $j \to \infty$ (convergence in norm)

(c) $\exists E \in \mathscr{M}$ such that $\mu(E) = 0$ and $\forall x \notin E$, $f_j(x) \to f(x)$ (convergence pointwise μ-a.e.)

(i) Show that (c)$\Rightarrow$ (b)$\Leftrightarrow$ (a).

(ii) What implications remain true if we now allow f, $f_j \colon X \to \mathbb{R}$ and/or $\mu(X) = \infty$?

(iii) Can you provide counterexamples to demonstrate the implications which do not hold in all the above cases? (It is possible to do so using only Lebesgue measure on $\mathbb{R}$ or $[0, 1]$).

Exercise 3

Let $(X, \mathscr{M}, \mu)$ be a measure space. Let f, $f_j \colon X \to \mathbb{R}$ be a sequence of measurable functions. Show that we have pointwise convergence μ-a.e. (in the sense of (c) in Ex. 2) provided we have the following, stronger version of convergence in measure:

$$\mu(\{x : |f(x) - f_k(x)| > 1/k\}) < 2^{-k} \quad \text{for all } k\,.$$

Hence, or otherwise, prove that convergence in measure in the sense of (a) in Ex. 2 implies that there exists a subsequence $j_k \nearrow \infty$ such that $f_{j_k} \to f$ pointwise μ-a.e.