PDE II
(Zentralübung)

Problem Sheet 9

In the following, U always denotes an open, bounded subset of $\mathbb{R}^n$ with C^1-boundary, and L is a uniformly elliptic partial differential operator of second order in divergence form,

$$Lu = -\sum_{i,j=1}^{n} \left(a^{ij} u_{x_i}\right)_{x_j} + \sum_{i=1}^{n} b^i u_{x_i} + cu$$

with $a^{ij} \in C^1(\overline{U})$, $b^i, c \in L^\infty(U)$.

Question 1

Let $u \in H^1(\mathbb{R}^n)$ have compact support and be a weak solution to the semilinear PDE

$$-\Delta u + g(u) = f \quad \text{in } \mathbb{R}^n,$$

where $f \in L^2(\mathbb{R}^n)$ and $g : \mathbb{R} \to \mathbb{R}$ is smooth, with $g(0) = 0$, $g' \geq 0$, and g' uniformly bounded on $\mathbb{R}$. Prove $u \in H^2(\mathbb{R}^n)$.

Question 2

Let $M(U)$ denote the class of measurable (with respect to Lebesgue measure) real-valued functions $u : U \to \mathbb{R}$. Let $f : U \times \mathbb{R} \to \mathbb{R}$ satisfy the following condition:

$$\begin{cases} f(\cdot, t) \text{ is measurable for all } t \in \mathbb{R}, \\ f(x, \cdot) \text{ is continuous for a.e. } x \in U. \end{cases} \tag{1}$$

Definition. The *Nemitski operator* N associated to f is the map

$$N : M(U) \to M(U), \quad (N(u))(x) := f(x, u(x)), \quad x \in U.$$

(a) Prove that N is well-defined, i.e. $N(M(U)) \subset M(U)$.

(b) Let $\alpha, \beta \geq 1$. Suppose there exist $g \in L^\beta(U)$ and $c > 0$ such that

$$|f(x,t)| \leq g(x) + c|t|^{\frac{\alpha}{\beta}}, \quad (x,t) \in U \times \mathbb{R}.$$

Prove that $N : L^\alpha(U) \to L^\beta(U)$ is continuous.

(c) Suppose there exist $g \in L^{\frac{2n}{n+2}}(U)$ and $c > 0$ such that

$$|f(x,t)| \leq g(x) + c|t|^p, \quad (x,t) \in U \times \mathbb{R}$$

for some $p < 2^* - 1$. Let

$$F(x,t) := \int_0^t f(x,s)\, ds.$$

Prove that if $u \in H_0^1(U)$, then $F(\cdot, u(\cdot)) \in L^1(U)$, and conclude that the map $\Phi : H_0^1(U) \to \mathbb{R}$ given by

$$\Phi(u) := \int_U F(x, u(x))\, dx$$

is well-defined.

Question 3

Let $a^{ij}, b^i, c \in C^\infty(U)$.

(a) Let $f \in L^2(U)$ and assume $u \in H^1(U)$ is a weak solution of $Lu = f$ in U. Prove that if $f \in C^\infty(V)$ for some $V \subset\subset U$, then $u \in C^\infty(V)$.

(b) Define the *singular support* of $u \colon U \to \mathbb{R}$ as

$$\operatorname{sing\,supp}(u) = U \setminus \{x \in U : \text{ there exits } r > 0 \text{ such that } u \in C^\infty(B_r(x))\}$$

Prove:

$$\operatorname{sing\,supp}(u) = \operatorname{sing\,supp}(Lu).$$

Question 4

Let $f \in L^2(U)$, and assume $u \in H_0^1(U)$ is a weak solution to

$$\begin{cases} Lu = f & \text{in } U, \\ u = 0 & \text{on } \partial U. \end{cases}$$

How little regularity of a^{ij}, b^i, c, and ∂U can you assume and still prove that u is a classical solution?