PDE II
(Zentralübung)

Problem Sheet 8

Question 1

Prove the following theorem:

Theorem. Let $U \subset \mathbb{R}^n$ be open and non-empty, and let $n \geq 2$. Then there exists $u, f \in C_c(U)$ such that u is a weak solution of $-\Delta u = f$, but $u \notin C^2(U)$.

Hint: For $U = B(0, 1/4)$, consider $u(x, y) = (x^2 - y^2) \ln |\ln \sqrt{x^2 + y^2}|$.

Question 2

Let H be a real Hilbert space.

(a) Prove the following lemma:

Lemma. Let $A : H \to H$ be a map satisfying

$$\|A(x) - A(y)\|_H \leq \gamma \|x - y\|_H, \quad x, y \in H, \tag{1}$$
$$(x - y, A(x) - A(y))_H \geq \alpha \|x - y\|_H^2, \quad x, y \in H, \tag{2}$$

for some $\alpha, \gamma > 0$. Then, for all $f \in H$ there exists a unique $u \in H$ such that $A(u) = f$. (*Hint:* Look at $R(v) := v - \lambda A(v) + \lambda f$, $\lambda \in \mathbb{R}$ and use a fixed point argument.)

(b) Prove the following theorem:

Theorem. Let $B : H \times H \to \mathbb{R}$ be a continuous map, linear in the second variable, and assume

$$|B[u_1, v] - B[u_2, v]| \leq \beta \|u_1 - u_2\|_H \|v\|_H, \quad u_1, u_2, v \in H, \tag{3}$$
$$B[u_1, u_1 - u_2] - B[u_2, u_1 - u_2] \geq C \|u_1 - u_2\|_H^2, \quad u_1, u_2 \in H, \tag{4}$$

for some $\beta, C > 0$. Then for every $f \in H'$, there exists a unique $u \in H$ such that $B[u, w] = \langle f, w \rangle$ for all $w \in H$.

Question 3

Prove the following theorem:

Theorem. Assume that $b^i = c = 0$, and let $g : \mathbb{R} \to \mathbb{R}$ be an increasing, globally Lipschitz-continuous function. Let $m \geq \frac{2n}{n+2}$. Then, for all $f \in L^m(U)$ there exists a unique weak solution $u \in H_0^1(U)$ to the semilinear Boundary Value Problem

$$\begin{cases} Lu + g(u) = f & \text{in } U, \\ u = 0 & \text{on } \partial U. \end{cases}$$

(*Hint:* Use Question 2.)

Question 4

Let $n = 3$ and $U = B(0, \pi) \subset \mathbb{R}^3$. Show that a necessary condition for the existence of a weak solution to

$$\begin{cases} -\Delta u - u = f & \text{in } U, \\ u = 0 & \text{on } \partial U, \end{cases}$$

is that

$$\int_U f(x) \frac{\sin(|x|)}{|x|} \, dx = 0.$$