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PDE II
(Zentralübung)

Problem Sheet 7

Question 1

For $U \subset \mathbb{R}^n$ open (but not necessarily bounded) we call U *bounded in one direction* if there exist $\nu \in \mathbb{R}^n$, $\|\nu\| = 1$ and $d > 0$, such that

$$U \subset \{x \in \mathbb{R}^n | \, |x \cdot \nu| \leq d\}. \tag{1}$$

We denote $\partial_\nu := \nu \cdot Du$.

(a) Prove the following theorem:

Theorem (Poincaré Inequality) Let $U \subset \mathbb{R}^n$ be bounded in one direction, such that (1) holds. Then

$$\|u\|_{L^2(U)} \leq \sqrt{2} d \|\partial_\nu u\|_{L^2(U)}, \quad \text{for all } u \in H_0^1(U).$$

Furthermore, on $H_0^1(U)$ the norms $\|u\|_{H^1(U)}$ and $\|Du\|_{L^2(U)}$ are equivalent.

(b) Assume L is a uniformly elliptic second order PDO in divergence form with $a^{ij} = a^{ji} \in L^\infty(U)$, $b^i = 0$ $(i, j = 1, \dots, n)$ and $c \in L^\infty(U)$, with $c(x) \geq 0$ for a.e. $x \in U$, i.e.

$$Lu = -\sum_{i,j=1}^{n} \left(a^{ij} u_{x_i}\right)_{x_j} + cu\,. \tag{2}$$

Prove the following theorem:

Theorem Let $f \in H^{-1}(U)$. Assume

Either (i) U is bounded in one direction,

or (ii) $c(x) \geq c$ for a.e. $x \in U$, for some $c > 0$.

Then there exists a unique weak solution to

$$\begin{cases} Lu = f & \text{in } U, \\ u = 0 & \text{on } \partial U. \end{cases}$$

Question 2

Prove the following theorem:

Theorem Let $U \subset \mathbb{R}^n$ be any open set (not necessarily bounded) and let $f \in L^2(U)$. Let $V \subset H^1(U)$ be a closed subspace. Then there exists a unique $u \in V$ such that

$$\int_U \left(\sum_{i=1}^{n} u_{x_i} v_{x_i} + uv \right) dx = \int_U fv \, dx \quad \text{for all } v \in V.$$

(Note that, in particular, $V = H_0^1(U)$ or $V = H^1(U)$ are closed subspaces of $H^1(U)$.)

Question 3

Let $U \subset \mathbb{R}^n$ and let L be a uniformly elliptic second order PDO in divergence form. For $f \in H^{-1}(U)$, $g \in H^1(U)$ we say that $u \in H^1(U)$ is a weak solution to

$$\begin{cases} Lu = f & \text{in } U, \\ u = g & \text{on } \partial U, \end{cases} \quad \text{if and only if} \quad \begin{cases} B[u,v] = \langle f, v \rangle & \text{for all } v \in H_0^1(U), \\ u - g \in H_0^1(U). \end{cases} \tag{3}$$

Here $B[\cdot\, , \cdot\,]$ denotes the bilinear form associated to L. Under the same assumptions as in Question 1 (b) (including the assumption in the theorem), prove that there exists a unique weak solution to (3).

Question 4

Assume $U \subset \mathbb{R}^n$ is open and bounded, and that ∂U is C^1.

(a) Assume also that U is connected. A function $u \in H^1(U)$ is a weak solution to *Neumann's problem*

$$\begin{cases} -\Delta u = f & \text{in } U, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial U, \end{cases} \quad \text{if and only if} \quad \int_U Du \cdot Dv \, dx = \int_U fv \, dx, \quad \text{for all } v \in H^1(U).$$

Given $f \in L^2(U)$, prove that there exists a weak solution if and only if $\int_U f \, dx = 0$.

(b) Explain how to define $u \in H^1(U)$ to be a weak solution to Poisson's equation with *Robin boundary conditions*:

$$\begin{cases} -\Delta u = f & \text{in } U, \\ u + \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial U. \end{cases}$$

Discuss the existence and uniqueness of a weak solution for given $f \in L^2(U)$.