PDE II
(Zentralübung)

Problem Sheet 6

In the following, U always denotes an open, bounded subset of $\mathbb{R}^n$ with smooth boundary.

Also, let L be a uniformly elliptic second order PDO in divergence form, that is (with $a^{ij} = a^{ji}$, $i, j = 1, \ldots, n$)

$$Lu = -\sum_{i,j=1}^{n} \left(a^{ij} u_{x_i}\right)_{x_j} + \sum_{i=1}^{n} b^i u_{x_i} + cu\,. \tag{1}$$

Question 1

Let L be as in (1). Assume $a^{ij} \in C^1(\overline{U})$, $b^i, c \in C(\overline{U})$. Let $f \in C(\overline{U})$.

(a) Assume $u \in C^2(\overline{U})$ is a *classical* solution to

$$\begin{cases} Lu = f & \text{in } U\,, \\ u = 0 & \text{on } \partial U\,. \end{cases} \tag{2}$$

Prove that u is a *weak* solution to (2).

(b) Assume $u \in H_0^1(U) \cap C^2(\overline{U})$ is a *weak* solution to (2). Prove that u is a *classical* solution to (2).

(c) How much can one weaken the assumptions and still prove (a) and/or (b)?

Question 2

Let L be as in (1), with $b^i \equiv 0$, and $a^{ij}, c \in L^\infty(U)$. Let B be the corresponding bilinear form, and let $f \in L^2(U)$.

(a) Let $I : H_0^1(U) \to \mathbb{R}$ be defined by

$$I[u] := \tfrac{1}{2} B[u,u] - (f,u) \ \text{ for } u \in H_0^1(U)\,.$$

Assume there exists a minimiser $u_0 \in H_0^1(U)$ for I,that is,

$$I[u_0] = \inf_{u \in H_0^1(U)} I[u] = \min_{u \in H_0^1(U)} I[u]\,.$$

Prove that u_0 is a weak solution to (2).

(b) Let $g : \mathbb{R} \to \mathbb{R}$ be continuous and bounded, and let $G(s) := \int_0^s g(t)\, dt$. Let $J : H_0^1(U) \to \mathbb{R}$ be defined by

$$J[u] := I[u] - \int_U G(u)\, dx \quad \text{for } u \in H_0^1(U)\,.$$

Assume there exists a minimiser $u_0 \in H_0^1(U)$ for J. Prove that u_0 is a weak solution to the semilinear elliptic equation

$$\begin{cases} Lu = f + g(u) & \text{in } U\,, \\ u = 0 & \text{on } \partial U\,. \end{cases} \tag{3}$$

(Note: An example is the nonlinear Poisson equation $-\Delta u = g(u)$.)

Question 3

Let B be a continuous, coercive, bilinear form on a real Hilbert space H. Let $f, \ell \in H'$, $\ell \neq 0$. Set

$$V := \{h \in H \,|\, \langle \ell, h\rangle = 0\}\,.$$

(a) Prove that there exists a unique solution u to

$$\begin{cases} u \in V, \\ B[u,v] = \langle f, v\rangle \text{ for all } v \in V\,. \end{cases} \tag{4}$$

(b) Prove that there exists a unique $k \in \mathbb{R}$ such that u satisfies

$$B[u,v] = \langle f + k\ell, v\rangle \text{ for all } v \in H\,.$$

(Hint: If $h \in H$ with $\ell(h) = 1$ then $v - \ell(v)h \in V$ for all $v \in H$.)

(c) Prove that there exists a unique solution u_ℓ to

$$\begin{cases} u_\ell \in H, \\ B[v, u_\ell] = \langle \ell, v\rangle \text{ for all } v \in H\,, \end{cases}$$

and that $k = -\frac{\langle f, u_\ell\rangle}{\langle \ell, u_\ell\rangle}$.

(d) For $\ell_1, \ell_2, \ldots, \ell_p \in H'$, let

$$V := \{h \in H \,|\, \langle \ell_i, h\rangle = 0 \text{ for all } i = 1, \ldots, p\}\,.$$

What can be said about the solution u to (4) in this case?

Question 4

Let B be a continuous, coercive, bilinear, *symmetric* form on a real Hilbert space H. Let $f \in H'$, and let $u \in H$ be the unique solution to

$$B[u,v] = \langle f, v\rangle \text{ for all } v \in H\,.$$

Prove that u is the unique minimiser on H of the functional $J : H \to \mathbb{R}$ given by

$$J[v] := \frac{1}{2}B[u,v] - \langle f, v\rangle\,.$$