PDE II
(Zentralübung)

Problem Sheet 5

Question 1

Let $U \subset \mathbb{R}^n$ be open and bounded.

(a) Let $0 \leq \alpha < \beta \leq 1$. Prove that $C^{0,\beta}(\overline{U}) \subset\subset C^{0,\alpha}(\overline{U})$ (compact embedding).

(b) Assume in addition that ∂U is C^1. Prove that $W^{1,p}(U) \subset\subset L^p(U)$ for all $1 \leq p \leq \infty$. (*Hint:* Distinguish between the cases $p \in [1, n]$ and $p \in (n, \infty]$; in the latter case use Morrey's inequality and (a).)

Question 2

Notation: For a map u and a set U, let $(u)_U := \fint_U u\, dy$, i.e. the mean value of u over U.

(a) Prove the mean value version of the Poincaré Inequality: let $U \subset \mathbb{R}^n$ be a bounded, connected open set with C^1 boundary. Let $1 \leq p < \infty$. Then there exists a constant $C > 0$ depending on U, n and p, such that

$$\int_U \left|u(x) - (u)_U\right|^p \mathrm{d}x \leq C \int_U |Du(x)|^p \mathrm{d}x$$

for all $u \in W^{1,p}(U)$.

Hint: Use Rellich-Kondrachoff and argue by contradiction. Also note that since the inequality does not change if we add a constant to u, it suffices to prove it for u with $\fint_U u = 0$.

(b) Deduce the following corollary of part (a): there exists a constant $C > 0$, depending on n and p, such that

$$\int_{B(x,r)} \left|u(x) - (u)_{B(x,r)}\right|^p \mathrm{d}x \leq Cr^p \int_{B(x,r)} |Du(x)|^p \mathrm{d}x$$

for each ball $B(x,r) \subset \mathbb{R}^n$ and each function $u \in W^{1,p}(B(x,r))$.

Question 3

(a) Let $u(x) := \ln\ln(1 + |x|^{-1})$ for $x \in B(0,1)$. Prove that $u \notin L^\infty(B(0,1))$, but $u \in W^{1,n}(B(0,1))$. (Hence $W^{1,n}(B(0,1)$ is not continuously embedded in $L^\infty(B(0,1))$.)

(b) Let $\mathrm{BMO}(\mathbb{R}^n)$ be the space of all $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ such that the seminorm

$$[u]_{\mathrm{BMO}(\mathbb{R}^n)} := \sup_{B(x,r) \subset \mathbb{R}^n} \fint_{B(x,r)} |u - (u)_{B(x,r)}| \, dy$$

is finite. ($\mathrm{BMO}(\mathbb{R}^n)$ is called the space of functions of *bounded mean oscillation*.) Prove that $W^{1,n}(\mathbb{R}^n)$ is continuously embedded in $\mathrm{BMO}(\mathbb{R}^n)$. (*Hint:* Use Question 2 (b).)

(c) Let $\overline{u} = Eu \in W^{1,n}(\mathbb{R}^n)$ be an extension of u such that $\overline{u} = u$ in $B(0,1)$, $\overline{u}$ has compact support and $\|\overline{u}\|_{W^{1,n}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(\mathbb{R}^n)}$ (Why does $\overline{u}$ exist?). Show that $\overline{u} \in \mathrm{BMO}(\mathbb{R}^n)$.