MATHEMATISCHES INSTITUT
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PDE II
(Zentralübung)

Problem Sheet 4

In the following, $U \subset \mathbb{R}^n$ will always denote an open set.

Question 1

Let U be bounded, with a C^1 boundary. Show that a "typical" function $u \in L^p(U)$ $(1 \leq p < \infty)$ does not have a trace on ∂U. More precisely, prove there does not exist a bounded linear operator

$$T : L^p(U) \to L^p(\partial U)$$

such that $Tu = u|_{\partial U}$ whenever $u \in C(\overline{U}) \cap L^p(U)$.

Question 2

Assume that U is bounded and there exists a smooth vector field $\boldsymbol{\alpha}$ such that $\boldsymbol{\alpha} \cdot \boldsymbol{\nu} \geq 1$ along ∂U, where $\boldsymbol{\nu}$ as usual denotes the outward unit normal. Assume $1 \leq p < \infty$.

Apply the Gauss-Green Theorem to $\int_{\partial U} |u|^p \boldsymbol{\alpha} \cdot \boldsymbol{\nu} \, dS$, to derive a new proof of the trace inequality

$$\int_{\partial U} |u|^p \, dS \leq C \int_U |Du|^p + |u|^p \, dx$$

for all $u \in C^1(\overline{U})$.

Question 3

(a) Integrate by parts to prove

$$\|Du\|_{L^p} \leq C \|u\|_{L^p}^{1/2} \|D^2 u\|_{L^p}^{1/2}$$

for $2 \leq p < \infty$ and all $u \in C_c^\infty(U)$.

Hint: $\int_U |Du|^p \, dx = \sum_{i=1}^n \int_U u_{x_i} u_{x_i} |Du|^{p-2} \, dx$.

(b) Prove

$$\|Du\|_{L^{2p}} \leq C \|u\|_{L^\infty}^{1/2} \|D^2 u\|_{L^p}^{1/2}$$

for $1 \leq p < \infty$ and all $u \in C_c^\infty(U)$.

Question 4

Prove Remark 1.13 from the lectures (for notation, see the lecture notes): if ∂U is C^2, then E is a bounded linear operator $E\colon W^{2,p}(U) \to W^{2,p}(\mathbb{R}^n)$.