MATHEMATISCHES INSTITUT
DER UNIVERSITÄT MÜNCHEN
Prof. Dr. T. Sørensen / Dr. P. Soneji

PDE II
(Zentralübung)

Problem Sheet 3

In the following, $U \subset \mathbb{R}^n$ will always denote an open set.

Question 1

Let $1 \leq p \leq \infty$. For $h \in \mathbb{R}^n$, let $\tau_h \colon L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ be the translation by h, i.e. $(\tau_h u)(x) = u(x - h)$.

(a) Prove that τ_h is a bounded linear operator with norm 1.

(b) Assume $1 \leq p < \infty$. Prove that τ_h is strongly continuous, i.e. $\lim_{h\to 0} \|\tau_h u - u\|_{L^p(\mathbb{R}^n)} = 0$. Prove or disprove the claim in the case $p = \infty$.

(c) Prove or disprove: τ_h is uniformly continuous, i.e. $\lim_{h\to 0} \|\tau_h - I\| = 0$, where $\|\cdot\|$ is the operator norm in $L^p(\mathbb{R}^n)$ and $I \colon L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ is the identity.

Question 2

Assume that U is bounded, and $F \in C^1(\mathbb{R})$ with $F' \in L^\infty(\mathbb{R})$. Let $u \in W^{1,p}(U)$ for some $1 \leq p < \infty$, and set $v := F \circ u$ on U. Prove:

$$v \in W^{1,p}(U), \quad \text{and} \quad Dv = F'(u)Du\,.$$

Question 3

Assume $1 \leq p < \infty$ and U is bounded. Let $u \in W^{1,p}(U)$ and define

$$u^+ := \begin{cases} u & \text{a.e. on } \{u > 0\}\,, \\ 0 & \text{a.e. on } \{u \leq 0\}\,, \end{cases}$$

as well as $u^- := (-u)^+$ and $|u| := u^+ + u^-$. Prove the following:

(a) $u^+, u^- \in W^{1,p}(U)$, and

$$Du^+ := \begin{cases} Du & \text{a.e. on } \{u > 0\}\,, \\ 0 & \text{a.e. on } \{u \leq 0\}\,, \end{cases}$$

$$Du^- := \begin{cases} 0 & \text{a.e. on } \{u \geq 0\}\,, \\ -Du & \text{a.e. on } \{u < 0\}\,, \end{cases}$$

Hint: Prove first that $u^+(x) = \lim_{\epsilon\to 0} F_\epsilon(u(x))$ a.e. where

$$F_\epsilon(z) := \begin{cases} (z^2+\epsilon^2)^{1/2} - \epsilon & \text{if } z \geq 0\,, \\ 0 & \text{if } z < 0\,. \end{cases}$$

(b) $|u| \in W^{1,p}(U)$, and $D|u| = \operatorname{sgn}(u) Du$.

(c) $Du = 0$ a.e. on the set $\{u = 0\}$.

Question 4

Integrate by parts to prove the interpolation inequality:

$$\|Du\|_{L^2} \leq C\|u\|_{L^2}^{1/2}\|D^2u\|_{L^2}^{1/2}$$

for all $u \in C_c^\infty(U)$. Assume U is bounded, ∂U is smooth, and prove this inequality if $u \in H^2(U) \cap H_0^1(U)$.

(*Hint:* Take sequences $(v_j) \subset C_c^\infty(U)$ converging to u in $H_0^1(U)$ and $(w_j) \subset C^\infty(\overline{U})$ converging to u in $H^2(U)$.)