PDE II
(Zentralübung)

Problem Sheet 2

In the following, $U \subset \mathbb{R}^n$ will always denote an open set.

Question 1

Prove the following inequalities:

(a) Assume $1 \le s \le r \le t \le \infty$, $0 < \theta < 1$ and $\frac{1}{r} = \frac{\theta}{s} + \frac{1-\theta}{t}$. Suppose $u \in L^s(U) \cap L^t(U)$. Then $u \in L^r(U)$, and
$$\|u\|_{L^r(U)} \le \|u\|_{L^s(U)}^{\theta} \|u\|_{L^t(U)}^{1-\theta}.$$

(b) Assume $0 < \beta < \gamma \le 1$ and $u \in C^{0,\beta}(\overline{U}) \cap C^{0,1}(\overline{U})$. Then $u \in C^{0,\gamma}(\overline{U})$, and
$$\|u\|_{C^{0,\gamma}(\overline{U})} \le \|u\|_{C^{0,\beta}(\overline{U})}^{\frac{1-\gamma}{1-\beta}} \|u\|_{C^{0,1}(\overline{U})}^{\frac{\gamma-\beta}{1-\beta}}.$$

Question 2

Suppose that U is connected, $u \in L^1_{\mathrm{loc}}(U)$, and the weak derivatives $D^\alpha u$ exist for all $|\alpha| \le k$. Prove the following: if $D^\alpha u = 0$ in U for all $|\alpha| = k$, then u is a polynomial in U of degree at most $k-1$.

Question 3

(a) Let $n = 1$, and let $u\colon [a,b] \to \mathbb{R}$ be a measurable function. Prove that u is absolutely continuous on $[a,b]$ if and only if the weak derivative of u exists and belongs to $L^1([a,b])$. Moreover, if this is the case, then the weak derivative coincides with the classical derivative almost everywhere.

(b) Assume that $u \in W^{1,p}([a,b])$ for some $1 < p < \infty$. Prove that
$$|u(x) - u(y)| \le |x-y|^{1-1/p} \left(\int_a^b |u'|^p \, \mathrm{d}x \right)^{1/p}$$
for almost every $x, y \in [a,b]$.

(c) Under the assumption of (a), prove that for $\alpha := 1 - \frac{1}{p}$,
$$\|u\|_{C^{0,\alpha}([a,b])} \le |u(x_0)| + C\|u'\|_{L^p([a,b])}$$
for any $x_0 \in [a,b]$ with a constant C that is independent of x_0.

Hint: You may use without proof the following facts about absolutely continuous functions:

(i) $u\colon [a,b] \to \mathbb{R}$ is absolutely continuous if and only if there exists a function $v \in L^1([a,b])$ such that

$$u(x) = u(a) + \int_a^x v(y)\,\mathrm{d}y\,, \quad x \in [a,b]\,.$$

(ii) If $u\colon [a,b] \to \mathbb{R}$ is absolutely continuous, then it is differentiable almost everywhere, and its derivative belongs to $L^1([a,b])$.

Question 4

Assume U is bounded and $U \subset\subset \cup_{i=1}^N V_i$, where the sets V_i are open. Prove that there exist C^∞-functions ζ_i ($i = 1, \ldots, N$) such that

(i) $0 \le \zeta_i \le 1$

(ii) $\mathrm{supp}(\zeta_i) \subset V_i$ for $i = 1, \ldots, N$

(iii) $\sum_{i=1}^N \zeta_i = 1$ on U.

The functions $\{\zeta_i\}_{i=1}^N$ form a *partition of unity*.