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PDE II
(Zentralübung)

Problem Sheet 11

In the following, U always denotes an open subset of $\mathbb{R}^n$.

Question 1

(a) Let $u \in C^2(U)$ and assume $\Delta u(x) > 0$ for all $x \in U$. Prove that u cannot attain a (local) maximum in U. Conclude that if in addition U is bounded and $u \in C^2(U) \cap C(\overline{U})$, then $\max_{\overline{U}} u = \max_{\partial U} u$.

(b) Let U be bounded and let $u \in C^2(U) \cap C(\overline{U})$. Assume $\Delta u \geq 0$ for all $x \in U$. Prove that $\max_{\overline{U}} u = \max_{\partial U} u$.

(*Hint:* Consider $u_\epsilon(x) := u(x) + \epsilon |x|^2$ for $\epsilon > 0$, $x \in U$, and apply (a).)

(c) Let $u \in C^2(U)$ and let $b(x) = (b_1(x), \dots, b_n(x))$, with $b_i \in C(\overline{U})$, $i = 1, \dots, n$. Assume

$$(Lu)(x) := \Delta u(x) + b(x) \cdot Du(x) > 0 \quad \text{for all } x \in U.$$

Prove that u cannot attain a (local) maximum in U, and conclude that if in addition U is bounded and $u \in C^2(U) \cap C(\overline{U})$, then $\max_{\overline{U}} u = \max_{\partial U} u$.

(d) Now additionally let $c \in C(\overline{U})$ with $c(x) \leq 0$ for all $x \in U$. Assume

$$(Lu)(x) := \Delta u(x) + b(x) \cdot Du(x) + c(x)d(x) \geq 0 \quad \text{for all } x \in U.$$

Prove that if u has a *non-negative* maximum in $\overline{U}$, then u cannot attain this maximum in U unless it is constant in U.

Remark: (c) is Theorem 2.35, (d) is Theorem 2.38 for $a^{ij} = \delta_{ij}$.

Question 2

(a) Let A, B, be symmetric, positive semi-definite (real) $n \times n$ matrices. Prove that $\operatorname{tr}(AB) \geq 0$.

(b) Assume U is bounded. Let $a^{ij} \in C(\overline{U})$, $i, j = 1, \dots, n$, with the (real) matrix $A(x) := \big(a^{ij}(x)\big)_{i,j=1}^n$ symmetric and positive semi-definite for all $x \in U$.

Assume $\operatorname{tr}(A(x)) = \sum_{i=1}^n a^{ii}(x) > 0$ for all $x \in U$. Let $u \in C^2(U) \cap C(\overline{U})$ and assume

$$\operatorname{tr}\big(A(x) D^2 u(x)\big) = \sum_{i,j=1}^n a^{ij}(x) u_{x_i x_j}(x) \geq 0 \quad \text{for all } x \in U\,.$$

Prove that $\max_{\overline{U}} u = \max_{\partial U} u$.

Question 3

Suppose U is bounded and has C^1 boundary. Let $L = \sum_{i,j=1}^n \left(a^{ij}u_{x_i}\right)_{x_j}$ where $a^{ij} \in C^1(\overline{U})$, L is uniformly elliptic, and (a^{ij}) is symmetric. Assume the operator L, with zero boundary conditions, has eigenvalues $0 < \lambda_1 < \lambda_2 \leq \ldots$. Show

$$\lambda_k = \max_{S \in \Sigma_{k-1}} \min_{\substack{u \in S^\perp \\ \|u\|_{L^2}=1}} B[u,u] \quad (k = 1, 2, \ldots).$$

Here, Σ_{k-1} denotes the collection of $(k-1)$-dimensional subspaces of $H_0^1(U)$.

Question 4

Suppose U is bounded and has C^1 boundary. Let $u, \widehat{u} \in H_0^1(U)$ both be positive minimizers of the Dirichlet energy

$$I[w] := \int_U |Dw|^2 dx.$$

under the constraint $\|w\|_{L^2(U)} = 1$.

Suppose also that $u, \widehat{u} > 0$ within U. Follow the hints to give a new proof that $u = \widehat{u}$ in U.

(*Hint:* Define $w := \left(\frac{u^2+\widehat{u}^2}{2}\right)^{1/2}$, $s := \frac{u^2}{u^2+\widehat{u}^2}$ and $\eta := \frac{u^2+\widehat{u}^2}{2}$; show that

$$|Dw|^2 = \eta \left| s\frac{Du}{u} + (1-s)\frac{D\widehat{u}}{\widehat{u}} \right|^2.$$

Deduce

$$|Dw|^2 \leq \eta \left(s\left|\frac{Du}{u}\right|^2 + (1-s)\left|\frac{D\widehat{u}}{\widehat{u}}\right|^2 \right) = \frac{1}{2}|Du|^2 + \frac{1}{2}|D\widehat{u}|^2$$

and therefore $\frac{Du}{u} = \frac{D\widehat{u}}{\widehat{u}}$ almost everywhere.)