MATHEMATISCHES INSTITUT
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PDE II
(Zentralübung)

Problem Sheet 10

Question 1

For $r > 0$, let B_r denote the ball $\{y \in \mathbb{R}^n : |y| < r\} = B(0, r)$. Prove the following:

Theorem. There exists a number M such that: if $u \in H^1(B_R)$ is a weak solution of $-\Delta u = f$ in B_R, for some $R > 0$, where $f \in L^2(B_R)$, then for all $r \in (0, R)$,

$$\|Du\|_{L^2(B_r)} \leq M\Big((R-r)\|f\|_{L^2(B_R)} + \frac{1}{(R-r)}\|u\|_{L^2(B_R)}\Big).$$

Question 2

(a) Prove the following theorem:

Theorem. Let $p \geq 1$, $n < \lambda < n + p$, and let U be an open, bounded domain for which there exists some $\delta > 0$ with

$$|B(x_0, r) \cap \overline{U}| \geq \delta r^n \quad \text{for all } x_0 \in U,\ r > 0.$$

Then a function $u \in L^p(U)$ is contained in $C^{0,\gamma}(\overline{U})$ for $\gamma = \frac{\lambda - n}{p}$ (or in $C^{0,1}(\overline{U})$ in the case $\lambda = n + p$) if and only if there exists a constant $K < \infty$ with

$$\int_{B(x_0,r)\cap U} |u(x) - u_{B(x_0,r)}|^p dx \leq K^p r^\lambda \quad \text{for all } x_0 \in U,\ r > 0$$

(where $u_{B(x_0,r)}$ is the average of u over the ball $B(x_0, r)$, and we have extended u by 0 on $\mathbb{R}^n \setminus B(x_0, r)$).

(b) Prove the corollary:

Corollary. If for all $0 < r \leq R_0$ and all $x_0 \in U$, we have

$$\int_{B(x_0,r)} |u - u_{B(x_0,r)}|^p dx \leq c r^{n+p\gamma}$$

with constants c and $0 < \gamma < 1$, then u is locally γ-Hölder continuous in U (this means that u is γ-Hölder continuous in any $V \subset\subset U$).

(*Hint*: You may use Lebesgue's Differentiation Theorem without proof.)

Question 3

Let $u_\epsilon = \eta_\epsilon * u$ denote the ϵ-mollification of $u \in C(\mathbb{R}^n)$ (see Problem Sheet 1, Question 4). Recall that the γ-Hölder-seminorm of u is defined as

$$[u]_{C^{0,\gamma}(\overline{U})} := \sup_{\substack{x,y\in U \\ x\neq y}} \left\{ \frac{|u(x)-u(y)|}{|x-y|^\gamma} \right\},$$

where $\gamma \in (0,1]$, $U \subset \mathbb{R}^n$ open. Now define

$$H_x^\gamma[u;U] := \sup_{y\in U} \left\{ \frac{|u(x)-u(y)|}{|x-y|^\gamma} \right\}.$$

We say that u *is Hölder continuous at* x *with respect to* U *with exponent* γ if this is finite. Note that also

$$[u]_{C^{0,\gamma}(\overline{U})} = \sup_{x\in V} H_x^\gamma[u;U]\,.$$

(a) Prove the following:

Lemma. Let $u \in C^{0,\gamma}_{\text{loc}}(\mathbb{R}^n)$, $\gamma \in (0,1]$. Then

$$|u_\epsilon(x) - u(x)| \leq \epsilon^\gamma H_x^\gamma[u; B(x,\epsilon)]$$

for all $x \in \mathbb{R}^n$.

(b) Prove:

Lemma. Let $u \in C(\mathbb{R}^n)$. If

$$\sup_{\substack{y\in B(x,R) \\ 0<\epsilon\leq R}} \epsilon^{1-\gamma}|Du_\epsilon(y)| < +\infty$$

for some $\gamma \in (0,1]$, $R > 0$, then u is Hölder continuous at x with respect to $B(x,R)$ with exponent γ. Furthermore

$$H_x^\gamma[u; B(x,r)] \leq C \sup_{\substack{y\in B(x,R) \\ 0<\epsilon\leq R}} \epsilon^{1-\gamma}|Du_\epsilon(y)|$$

with $C = C(\gamma, n, \eta) > 0$.

(c) Prove:

Proposition. There exists $C = C(\gamma, n, \eta) > 0$ such that

$$\frac{1}{C}[u]_{C^{0,\gamma}(\mathbb{R}^n)} \leq \sup_{\substack{y\in\mathbb{R}^n \\ \epsilon>0}} \epsilon^{1-\gamma}|Du_\epsilon(y)| \leq C[u]_{C^{0,\gamma}(\mathbb{R}^n)}$$

for all $u \in C^{0,\gamma}(\mathbb{R}^n)$.

Remark. This characterisation of $C^{0,\gamma}$ (and $C^{k,\gamma}$) can be used to prove Schauder's estimate (see remark after Theorem 2.27 in the lectures).

Question 4

For $U \subset \mathbb{R}^n$ open, bounded, let $I\colon H_0^1(U) \to \mathbb{R}$ be defined by

$$I[u] := \int_U \frac{1}{2}|Du(x)|^2 - f(x)u(x)\,\mathrm{d}x$$

for $u \in H_0^1(U)$, where $f \in L^2(U)$. Prove that there exists a minimiser $u_0 \in H_0^1(U)$ of I, that is

$$I[u_0] = \inf_{u \in H_0^1(U)} I[u] = \min_{u \in H_0^1(U)} I[u]\,.$$

(Hint: Recall from Functional Analysis that a bounded sequence $(u_j) \subset H_0^1(U)$ has a weakly convergent subsequence).