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PDE II
(Zentralübung)

Problem Sheet 1

In the following, $U \subset \mathbb{R}^n$ will always denote an open set.

Definition: for $k \geq 1$, $1 \leq p \leq \infty$, the *Sobolev Space* $W^{k,p}(U)$ is defined to be the space of functions $u \in L^p(U)$ whose weak partial derivatives $D^\alpha u$ exist and are also in $L^p(U)$, for all multi-indices α with $|\alpha| \leq k$.

Question 1

(a) Let $n = 1$ and $U = (0, 2)$. Compute the weak derivative (if it exists) of the following L^1_{loc} functions.

$$u(x) := \begin{cases} x & \text{if } 0 < x \leq 1 \\ 1 & \text{if } 1 < x < 2\,, \end{cases} \qquad v(x) := \begin{cases} x & \text{if } 0 < x \leq 1 \\ 2 & \text{if } 1 < x < 2\,. \end{cases}$$

(b) Let $U = B(0, 1)$ be the unit ball in $\mathbb{R}^n$, and

$$u(x) := |x|^{-\alpha} \quad (\, x \in U\,,\, x \neq 0\,).$$

For which values of $\alpha > 0$, n, and $p \geq 1$ does u belong to $W^{1,p}(U)$?

Question 2

Assume that u, $v \in W^{k,p}(U)$ and $|\alpha| \leq k$. Prove the following statements:

1. $D^\alpha u \in W^{k-|\alpha|,p}(U)$ and $D^\beta(D^\alpha u) = D^\alpha(D^\beta u) = D^{\alpha+\beta} u$ for all multi-indices α, β with $|\alpha| + |\beta| \leq k$.
2. For each λ, $\mu \in \mathbb{R}$, $\lambda u + \mu v \in W^{k,p}(U)$ and $D^\alpha(\lambda u + \mu v) = \lambda D^\alpha u + \mu D^\alpha v$ for $|\alpha| \leq k$.
3. If V is an open subset of U, then $u \in W^{k,p}(U)$.

Question 3

Prove the *Fundamental Lemma of the Calculus of Variations*: if $u \in L^1_{\text{loc}}(U)$ and

$$\int_U u\varphi \, \mathrm{d}x = 0 \quad \text{for all } \varphi \in C_c^\infty(U)\,,$$

then $u = 0$ almost everywhere in U. (You may use the result from question 4 (c)).

Question 4

Let $\eta \in C_c^\infty(\mathbb{R}^n)$ satisfy

(i) $\eta \geq 0$, $\eta(x) = 0$ if $|x| \geq 1$ (so $\mathrm{supp}(\eta) \subset\subset B(0,1)$).

(ii) $\int_{\mathbb{R}^n} \eta \,\mathrm{d}x = 1$.

Then, for $\epsilon > 0$, define $\eta_\epsilon(x) := \epsilon^{-n}\eta(x/\epsilon)$. Recall from PDE I that η_ϵ is said to be a *mollifier*. Recall also that for a function u we define the convolution

$$(\eta_\epsilon * u)(x) := \int_{\mathbb{R}^n} \eta_\epsilon(x-y)u(y)\,\mathrm{d}y\,.$$

(a) Let $1 \leq p \leq \infty$ and $u \in L^p(U)$. Define u to be zero outside U. Show that $\eta_\epsilon * u \in C^\infty(\mathbb{R}^n)$. (*Hint:* use difference-quotients, the dominated convergence theorem, and induction).

(b) Now suppose $u \in C_c(U)$. Show that $\eta_\epsilon * u$ converges uniformly to u as $\epsilon \to 0$.

(c) Now let $1 \leq p < \infty$ and $u \in L^p(U)$, with u zero outside U. Show that $\|\eta_\epsilon * u - u\|_p \to 0$ as $\epsilon \to 0$. (And hence $C^\infty(U)$ is dense in $L^p(U)$). You may use without proof the following facts:

   (i) $C_c(U)$ is dense in $L^p(U)$ for $1 \leq p < \infty$.

   (ii) $\|\eta_\epsilon * u\|_p \leq \|u\|_p$.