On Kaplansky's conjectures

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Abstract

We survey the known results on Kaplansky's ten conjectures on Hopf algebras.

1 Introduction

In the autumn of 1973, I. Kaplansky gave a course on bialgebras in Chicago. For this course, he prepared some lecture notes that he originally intended to turn into a comprehensive account on the subject. In 1975, he changed his mind and published these lecture notes without larger additions. These lecture notes contain, besides a fairly comprehensive bibliography of the literature available at that time, two appendices. The first of these appendices is concerned with bialgebras of low dimension, whereas the second one contains a list of ten conjectures on Hopf algebras which are known today as Kaplansky's conjectures.

Kaplansky's conjectures did not arise as the product of a long investigation in the field of Hopf algebras; also, Kaplansky did not make many contributions to the solution of his conjectures. He only intended to list a number of interesting problems at the end of his lecture notes - lecture notes that he himself called informal. Because of this, it happened that one conjecture in the list was already solved at the time of publication; another one is very simple. That the conjectures nevertheless gained considerable importance for the field is due to the fact that Kaplansky achieved to touch upon a number of questions of fundamental character.

This article tries to summarize the present knowledge about Kaplansky's conjectures. Brief surveys can be found in [66] and [78], some conjectures are also discussed in [50] and [56]. Here, the exposition shall be more detailed, but nevertheless not comprehensive. Since Kaplansky's lecture notes are not always easily accessible, we have reproduced the conjectures in their original formulation in an appendix. The reader should note that, except for the appendix, the formulation of results does not follow literally the quoted sources. In addition, usually not all important results of a quoted article are mentioned. We also note that Kaplansky posed various other conjectures concerning different fields of mathematics; these are not discussed here.

2 The first conjecture

2.1 Kaplansky's first conjecture states that a Hopf algebra is a free module over any Hopf subalgebra. This assertion should be compared with the group ring situation, where a system of representatives for the cosets forms a basis of the group ring over the group ring of a subgroup. At the time of its publication, it was already known that the conjecture is false, because U. Oberst and H.-J. Schneider had, in an article quoted by Kaplansky, constructed a counterexample (cf. [64], Prop. 10, p. 31, see also [50], Example 3.5.2, p. 38). In this counterexample, Oberst and Schneider construct an extension of real Hopf algebras via Galois descent from the extension

$$\mathbb{C}[2\mathbb{Z}] \subset \mathbb{C}[\mathbb{Z}]$$

of the complex group ring of the integers over the group ring of the even numbers. In fact, instead of the extension $\mathbb{R} \subset \mathbb{C}$, they consider more generally arbitrary quadratic Galois extensions of fields. Later, Schneider gave a counterexample over arbitrary fields (cf. [79], Example 4.6, p. 157): Suppose that H is the Hopf algebra representing the affine group scheme of the special linear group of degree n and $A = K[\mathbb{Z}_n]$ the group ring of the cyclic group of order n, representing the affine group scheme of n-th roots of unity. The roots of unity are realized via diagonal matrices as a normal subgroup of the special linear group. In this situation, we can look at the quotient scheme, which is represented by a Hopf algebra B. Now Schneider proves that H is not free over B if n is even.

2.2 However, several positive results were proved in the following years, showing that the conjecture holds under additional assumptions. Kaplansky already mentioned that W. D. Nichols proved that the conjecture holds if the coradical of the large Hopf algebra is contained in the Hopf subalgebra; this is also a direct consequence of a result by D. E. Radford (cf. [71], Cor. 2.3, p. 146). In the same year, Radford proved that the conjecture holds for a large class of Hopf algebras, namely the pointed ones (cf. [70], p. 271):

Theorem Suppose that A is a Hopf subalgebra of the Hopf algebra H. Suppose that H is pointed, i. e., every simple subcoalgebra is one-dimensional. Then H is a free left and right A-module.

Furthermore, Radford proved in [72] that commutative Hopf algebras are free over finite-dimensional Hopf subalgebras. He also proved that here the assumption that the Hopf subalgebra be finite-dimensional could be weakened to the assumption that only the coradical of the Hopf subalgebra be finite-dimensional by imposing the additional assumption that the coradical of the large Hopf algebra is a Hopf subalgebra, which is the case over fields of characteristic zero by a result of M. Takeuchi (cf. [95], Prop. 0, p. 453). He also proved there that

Hopf algebras are free over finite-dimensional Hopf subalgebras if the coradical of the large Hopf algebra is cocommutative.

2.3 W. D. Nichols had been concerned with Kaplansky's conjectures from the very beginning. In the eighties, he and his former Ph. D. student M. B. Zoeller worked in particular on Kaplansky's first conjecture. They summarized their efforts in a series of articles of subsequent generalizations (cf. [106], [57], [59]), culminating in what is now known as the Nichols-Zoeller theorem:

Theorem A finite-dimensional Hopf algebra is free over every Hopf subalgebra.

In fact, Nichols and Zoeller prove the more general theorem that relative Hopf modules are free. Their theorem implies that the dimension of a Hopf subalgebra divides the dimension of the large Hopf algebra, which is the Hopf algebra version of Lagrange's theorem on the order of subgroups. This theorem has many consequences and is commonly considered as one of the most important theorems of the theory of Hopf algebras. Nichols and Zoeller also exhibited an example that infinite dimensional Hopf algebras need not be free over finite dimensional grouplike Hopf subalgebras (cf. [58]). These authors have also shown in a more recent article that a Hopf algebra is free over a semisimple Hopf subalgebra (cf. [60]). (Due to marriage, the name 'Zoeller' has changed to 'Richmond'.)

The work of Nichols and Zoeller has been amplified by H.-J. Schneider. Given a Hopf subalgebra of a finite-dimensional Hopf algebra, it is possible to form a quotient which is not always a Hopf algebra, but always is a coalgebra. The original Hopf algebra then becomes a comodule over this quotient. Now Schneider shows (cf. [80], Thm. 2.4, p. 300) that the original Hopf algebra can be decomposed, as a module and a comodule, into a tensor product of the Hopf subalgebra and the quotient. In a different article, Schneider proves that Hopf algebras are free over finite dimensional normal Hopf subalgebras (cf. [81], Thm. 2.1, p. 3345).

The Nichols-Zoeller theorem has been extended to coideal subalgebras that are quasi-Frobenius algebras by A. Masuoka (cf. [41]). In this article, he also determines sufficient conditions for a coideal subalgebra to be a quasi-Frobenius algebra (cf. also [42]).

2.4 Although nowadays quite a lot is known about Kaplansky's first conjecture, the fact that it does not hold in the infinite-dimensional case has given rise to the question whether weaker properties than freeness hold in all cases. One possible variant of this question has been stated by S. Montgomery in [50], Question 3.5.4, p. 39:

Is H always left and right faithfully flat over any subHopfalgebra K?

This question might be considered as a resumption of Kaplansky's first conjecture. It plays an important rôle for the quotient theory of Hopf algebras.

Several things are known about this question today. It has been known for quite a time from the theory of algebraic groups that this is the case if the Hopf algebra is commutative (cf. [16], Chap. III, § 3, no. 7, Thm. 7.2, [100], Sec. 14.1). This result can also be found in an article by M. Takeuchi (cf. [94], Thm. 3.1), where it is also shown that a Hopf algebra with cocommutative coradical is faithfully flat over every Hopf subalgebra. Later, Takeuchi proved that commutative Hopf algebras are projective over their Hopf subalgebras (cf. [97], Cor. 1, p. 460). However, as was shown later by A. Masuoka and D. Wigner (cf. [43]), faithful flatness and projectivity are essentially equivalent conditions in this situation:

Theorem Suppose that H is a Hopf algebra with bijective antipode, and that A is a right coideal subalgebra of H. Then the following assertions are equivalent:

1. H is faithfully flat as a left A-module.
2. H is a projective generator as a left A-module.

This theorem applies in particular if A is a Hopf subalgebra of H. Of course, the first statement can easily be deduced from the second. In addition, H.-J. Schneider has proved that a left noetherian Hopf algebra is faithfully flat over central Hopf subalgebras (cf. [81], Thm. 3.3, p. 3355). S. H. Ng has shown that Hopf algebras are faithfully flat over grouplike Hopf subalgebras if the characteristic of the base field is zero (cf. [54]); he also considers the case of positive characteristic.

These results have been partially generalized to coideal subalgebras: A. Masuoka and D. Wigner prove that commutative Hopf algebras are flat over right coideal subalgebras (cf. [43]); they are not faithfully flat in general. Moreover, Masuoka has shown that Hopf algebras with cocommutative coradical are faithfully flat over certain coideal subalgebras (cf. [40]).

However, the answer to the above question is negative in general: P. Schauenburg has constructed a counterexample (cf. [76]) by modifying constructions of W. D. Nichols and M. Takeuchi (cf. [55], [96]). His example relies on the fact that the Hopf subalgebra can be chosen to have a non-bijective antipode, although the antipode of the large Hopf algebra may be bijective. Therefore, as Schauenburg points out, the remaining question is whether Hopf algebras with bijective antipodes are faithfully flat over Hopf subalgebras with bijective antipodes. This question may be considered as the current form of Kaplansky's first conjecture.

3 The second and the fourth conjecture

3.1 Kaplansky's second conjecture states that a coalgebra is admissible if and only if every finite subset is contained in a finite-dimensional admissible subcoalgebra. Here, a coalgebra is called admissible if it admits an algebra structure making it a Hopf algebra. This question should be viewed as a variant of the theorem that every finite subset of a coalgebra is contained in a finite-dimensional subcoalgebra (cf. [28], Thm. 2, p. 7, [50], Thm. 5.1.1, p. 56). R. G. Larson has pointed out the following simple counterexample to this conjecture: The polynomial ring $K[x]$ in one indeterminate over a field of characteristic zero becomes a Hopf algebra if x is required to be primitive. However, x is not contained in a finite-dimensional admissible subcoalgebra. Namely, if H is an admissible subcoalgebra containing x, it also contains the unit of $K[x]$ as the unit of its new algebra structure, because $K[x]$ is connected (cf. [50], Prop. 5.5.3, p. 73). It is then easy to see that the algebra homomorphism from $K[x]$ to H that maps x to itself is a coalgebra homomorphism. Because the characteristic of the base field is zero, the first step of the coradical filtration of the polynomial ring is spanned by 1 and x (cf. [50], Prop. 5.5.3, p. 74), and therefore the Hopf algebra homomorphism is injective on this subspace. Therefore, it is injective on the whole polynomial ring (cf. [50], Thm. 5.3.1, p. 65), and H must be infinite-dimensional.

3.2 Kaplansky's fourth conjecture states that, if an element x in a Hopf algebra H satisfies

$$a_{(1)}xS(a_{(2)}) = \epsilon(a)x$$

for every $a \in H$, then x is contained in the center of H. Here, we have used the following variant of the Heyneman-Sweedler sigma notation for the coproduct:

$$\Delta(a) = a_{(1)} \otimes a_{(2)}$$

This is obvious, because then we have:

$$ax = a_{(1)}xS(a_{(2)})a_{(3)} = \epsilon(a_{(1)})xa_{(2)} = xa$$

This calculation can certainly be found in many places, among them [50], Lem. 5.7.2, p. 83.

4 The third and the ninth conjecture

4.1 Kaplansky's third conjecture states that a Hopf algebra over a field of characteristic zero does not contain nonzero central nilpotent elements. Although it seems that no literature on this conjecture exists, it has been obvious to most researchers for some time that this is false.

Consider a left integral Λ_H in a finite-dimensional non-semisimple Hopf algebra. By the generalized Maschke theorem due to R. G. Larson and M. E. Sweedler (cf. [38], Prop. 3, p. 84), we then have $\epsilon_H(\Lambda_H) = 0$, and therefore also $\Lambda_H^2 = 0$. Λ_H is therefore a nilpotent element. Now, if Λ_H is also a right integral, i. e., if H is unimodular, then Λ_H is obviously a central element. Kaplansky's third conjecture would therefore imply that, over fields of characteristic zero, all finite-dimensional unimodular Hopf algebras are semisimple.

This is, however, not the case. There exist finite-dimensional Hopf algebras that are not semisimple, for example the Taft algebra (cf. [92]), which is, however, not unimodular. But every finite-dimensional Hopf algebra may be embedded in its Drinfel'd double $D(H)$, and this is a unimodular Hopf algebra by a result of D. E. Radford (cf. [74], Thm. 4, p. 303, [85], Thm. 5.4, p. 62). By another result of Radford (cf. [74], Prop. 7, p. 304), $D(H)$ is semisimple if and only if H is semisimple and cosemisimple. Therefore, the Drinfel'd double of the Taft algebra is a finite-dimensional unimodular Hopf algebra which is not semisimple.

4.2 We now consider another, related counterexample to Kaplansky's third conjecture, namely the Frobenius-Lusztig kernel of $U_q(sl(2))$, which is slightly more explicit than the example considered above and, as we shall see below, also provides a counterexample to Kaplansky's ninth conjecture. In fact, the Frobenius-Lusztig kernel of $U_q(sl(2))$ can be obtained as a quotient of the Drinfel'd double of the Taft algebra.

We work over an algebraically closed field of characteristic zero. Suppose that q is a primitive k-th root of unity, where $k > 2$ is odd - the even case is similar. The Frobenius-Lusztig kernel of $U_q(sl(2))$ is defined as the algebra U with generators K, K^{-1}, E, and F and relations

$$KK^{-1} = 1 = K^{-1}K$$
$$KE = q^2EK \qquad KF = q^{-2}FK$$
$$EF - FE = \frac{K - K^{-1}}{q - q^{-1}}$$
$$K^k = 1 \qquad E^k = 0 \qquad F^k = 0$$

The coproduct is determined on the generators by the formulas:

$$\Delta(K) = K \otimes K \quad \Delta(K^{-1}) = K^{-1} \otimes K^{-1}$$
$$\Delta(E) = E \otimes 1 + K \otimes E \quad \Delta(F) = F \otimes K^{-1} + 1 \otimes F$$

This Hopf algebra has a basis consisting of the elements $E^iK^jF^m$ for $i, j, m = 0, \ldots, k-1$. If $T := \sum_{i=1}^{k} q^{2i(k-1)}K^i$, it is easy to see that $E^{k-1}TF^{k-1}$ is a left and right integral, and therefore a central element (cf. [89], p. 368, [85], Prop. 6.3, p. 67, note a slight difference in the definitions). Since the counit vanishes on the integral, its square is zero.

The integral is not the only nilpotent central element of U. Its center can be described completely (cf. [77]).

4.3 As noted by N. Andruskiewitsch and S. Natale in an appendix to notes by H.-J. Schneider (cf. [78]), the Frobenius-Lusztig kernel considered above also provides a counterexample to Kaplansky's ninth conjecture. This conjecture states that for a finite-dimensional Hopf algebra over an algebraically closed field, the dimension of the Jacobson radical of the Hopf algebra equals the dimension of the Jacobson radical of the dual if the characteristic of the base field does not divide the dimension. In particular, semisimple Hopf algebras should be cosemisimple, a problem that will be considered below in conjunction with the fifth conjecture.

We now explain briefly why Kaplansky's ninth conjecture is refuted by the Frobenius-Lusztig kernel U. The simple modules of U are known (cf. [29], Thm. VI.5.7, p. 137, [26], Sec. 2.13, p. 24); in the odd case considered above, there is one isomorphism class of simple modules for each of the dimensions $1, \ldots, k$. Since the Jacobson radical J of U consists, by definition, precisely of the elements that annihilate every simple module, U/J has the same simple modules as U. Since U/J is a semisimple algebra, its dimension is the sum of the squares of the dimensions of the simple modules, i. e., we have:

$$\dim U/J = \sum_{m=1}^{k} m^2 = \frac{1}{6} k(k+1)(2k+1)$$

On the other hand, the Jacobson radical of U^* is precisely orthogonal to the coradical C of U (cf. [50], Rem. 5.1.7, p. 58). Kaplansky's conjecture therefore would imply that the coradical has the same dimension as U/J. But, as can be shown as in [50], Lem. 5.5.5, p. 76, the coradical of U is spanned by the powers of K, and therefore we have:

$$\dim C = k$$

Note that the proof in [50] needs a slight modification. Further details on the Jacobson radical of U can be found in [89], Cor. 3.8, p. 367; note that the setup there is slightly different.

5 The fifth and the seventh conjecture

5.1 It is easy to see that the antipode of a Hopf algebra is an involution if the Hopf algebra is commutative or cocommutative. Since the notion of an antipode in a Hopf algebra generalizes the notion of an inverse in a group, it is reasonable to expect that this is a more general feature. In fact, it was not known for some time whether there exist Hopf algebras whose antipodes were not involutions - at least in the finite-dimensional case. Such a Hopf algebra was then constructed by E. J. Taft (cf. [92], [93]), and also by M. E. Sweedler (unpublished) and D. E. Radford (cf. [67]). Before that, R. G. Larson as well as M. E. Sweedler already had constructed examples of infinite-dimensional Hopf algebras with antipodes of infinite order (cf. [31], [91], Chap. IV, p. 89), whereas Radford proved later that antipodes of finite-dimensional Hopf algebras have finite order (cf. [68], [69]).

When Kaplansky wrote down his conjectures, it was therefore known that the antipode of a Hopf algebra is not always an involution. Kaplansky's fifth conjecture now states that the antipode of a finite-dimensional Hopf algebra over an algebraically closed field is an involution if the Hopf algebra is semisimple or cosemisimple. Here a Hopf algebra is called cosemisimple if the dual Hopf algebra is semisimple.

It should first be observed that the question of algebraic closure is not important here, because the order of the antipode remains unchanged under base field extension. As will be explained below, the characteristic of the base field is of greater importance.

One of the first results on this conjecture was obtained by R. G. Larson, who proved it for Hopf algebras over fields of characteristic zero or sufficiently large characteristic, provided that the Hopf algebra is semisimple and the irreducible modules have dimension one or two (cf. [34]). His results were generalized by D. E. Radford, who proved that it suffices to assume that the two-sided ideals corresponding to the simple modules of dimension one or two generate the Hopf algebra H as a coalgebra (cf. [73]).

Kaplansky's fifth conjecture is better understood if it is split into two partial problems:

1. Is every semisimple Hopf algebra cosemisimple?
2. Is the antipode of a semisimple, cosemisimple Hopf algebra an involution?

In what follows, we shall treat these problems separately.

5.2 An important step towards Kaplansky's fifth conjecture was taken by R. G. Larson and D. E. Radford in two closely related articles published in 1988. In the first one of these they answered affirmatively the first question above under the assumption that the base field has characteristic zero (cf. [36]). Their proof relies on a trace formula that expresses the trace of the squared antipode in terms of the integrals of the Hopf algebra:

$$\mathrm{Tr}(S_H^2) = \epsilon_H(\Gamma_H)\rho_H(1)$$

Here, ϵ_H and S_H denote the counit and the antipode of the Hopf algebra H, whereas $\Gamma_H \in H$ and $\rho_H \in H^*$ denote right integrals satisfying $\rho_H(\Gamma_H) = 1$. As H.-J. Schneider points out, the immediate consequence $\dim H = \epsilon_H(\Gamma_H)\rho_H(1)$ if $S_H^2 = \mathrm{id}_H$ was already observed by U. Oberst and himself (cf. [63], Satz 3.6, p. 15). He has later used the methods of this article to give simplified proofs of the first trace formula stated above as well as the second trace formula discussed below, and also for Radford's formula for the fourth power of the antipode (cf. [78]).

A result on the first problem mentioned above in large positive characteristic was proved in 1997 (cf. [84]): It was shown that, if characteristic p of the base field is zero or satisfies $p > m^{m-4}$, where $m := 2\,(\dim H)^2$, then H is cosemisimple. The approach used there is rather different from the approach used by Larson and Radford, and can be roughly broken into two steps: The first step was the adjunction of a grouplike element that made the square of the antipode an inner automorphism. The trace of the squared antipode is then the value of the character of the adjoint representation on that grouplike element. Now, if this character is invertible, this value is nonzero, and therefore the first trace formula stated above implies that the Hopf algebra is also cosemisimple. Then, in the second step, the invertibility of the character of the adjoint representation was established by using the Perron-Frobenius theorem to calculate the largest eigenvalue of the left multiplication with this character.

This bound on the characteristic has been improved by P. Etingof and S. Gelaki in the final version of [20]. They prove that a semisimple Hopf algebra H of dimension bigger than two is cosemisimple if $p > n^{\varphi(n)/2}$, where $n := \dim H$ is the dimension of H and φ is the Euler φ-function (cf. [20], Thm. 4.2, p. 862). Their argument reduces the problem to the characteristic zero case by using the ring of Witt vectors of the base field (cf. [102]). They then can essentially follow the original argument by Larson and Radford. Further results from this important contribution will be described below.

5.3 We now discuss the second problem mentioned above. It should be observed that for this problem, we already know by Radford's formula that the fourth power of the antipode is the identity (cf. [69], Prop. 6, p. 347). In their second article, which was also published in 1988, Larson and Radford also made important progress on the second problem. There, they proved that the antipode of a semisimple, cosemisimple Hopf algebra is an involution if the characteristic of the base field is zero or larger than the square of the dimension of the Hopf algebra (cf. [35], Thm. 3, p. 194). Their technique relies on a second trace formula:

$$\mathrm{Tr}(S_H^2) = (\dim H)\,\mathrm{Tr}(S_H^2\,|_{x_R H})$$

Here, $x_R \in H$ denotes the character of the regular representation of H^*, i. e., the element satisfying $\phi(x_R) = \mathrm{Tr}(L_\phi)$, where $L_\phi : H^* \to H^*$ denotes the left multiplication by ϕ. It should be noted that it is reasonable to expect that this character is an integral of H. If it were possible to prove this in general, the above trace formula would read $\mathrm{Tr}(S_H^2) = \dim H$, and it would be easy to derive strong conclusions on Kaplansky's fifth conjecture.

In their article already mentioned above, P. Etingof and S. Gelaki have settled the second problem completely: They prove that the antipode of a semisimple, cosemisimple Hopf algebra over any field is an involution (cf. [20], Thm. 3.1, p. 858). Their method, which has several other applications that will be discussed below, is based on a cohomology vanishing result of D. Ştefan (cf. [87]) that was originally invented to deal with Kaplansky's tenth conjecture, which will also be discussed below. Etingof and Gelaki use this result to prove that, for a semisimple, cosemisimple Hopf algebra, there exists a Hopf algebra that is defined over a complete discrete valuation ring of characteristic zero, the ring of Witt vectors, such that the original Hopf algebra can be obtained from this Hopf algebra by reduction modulo the maximal ideal of the discrete valuation ring. By instead passing to the quotient field of this discrete valuation ring, they can apply the characteristic zero result to prove that the antipode is an involution.

5.4 The methods of Larson and Radford described above also enabled them to solve Kaplansky's seventh conjecture. This conjecture states that for a semisimple, cosemisimple Hopf algebra over an algebraically closed field, the characteristic of the field does not divide the dimension of the Hopf algebra. Once again, the assumption that the base field be algebraically closed is not necessary. Using the above trace formulas, it is possible to give their complete proof of this conjecture in a few lines (cf. [35], Thm. 2, p. 194):

By Larson and Sweedler's generalization of Maschke's theorem, a Hopf algebra is semisimple if and only if the counit does not vanish on a nonzero integral. Therefore, the right hand side of the first trace formula is nonzero if the Hopf algebra is semisimple and cosemisimple. Therefore, also the left hand side is nonzero, and by the second trace formula this implies that the dimension of the Hopf algebra is nonzero as an element of the base field, i. e., the characteristic of the base field does not divide the dimension.

Probably not aware of Kaplansky's precise formulation, Etingof and Gelaki also claim in their article to settle Kaplansky's seventh conjecture (cf. [20], p. 858); they also quote the result of Larson and Radford in their proof.

5.5 There are also further results on this conjecture. According to W. D. Nichols, M. Eberwein has proved that the antipode of a semisimple Hopf algebra is an involution if that the characteristic of the base field is larger than the squared dimension of the Hopf algebra and the irreducible modules are of dimension 1, 2, or 3 (cf. [17], Thm. 4.3.11, [56], Thm. 5.9, p. 150). This generalizes the above-mentioned result of Larson (cf. [34]). Larson and Radford have also proved a number of additional results: Among other things, they have studied the question whether the property that the antipode is an involution can be extended from a Hopf subalgebra to a larger Hopf algebra, and they have carried out computer-based analysis of the algebra structures for a potential counterexample to Kaplansky's fifth conjecture (cf. [37]).

6 The sixth conjecture

6.1 Kaplansky's sixth conjecture may be interpreted as follows: For a finite-dimensional Hopf algebra over an algebraically closed field, the dimension of every simple module divides the dimension of the Hopf algebra. In this generality, it was known at that time that this is false even for group rings. For example, the special linear group $SL(2,p)$ of 2×2-matrices over a field with p elements, where p is an odd prime, admits simple modules over algebraically closed fields of characteristic p with a dimension that does not divide the order of the group (cf. [15], Example (17.17), p. 426). However, it was an open question at that time whether, for an absolutely irreducible representation in characteristic $p > 0$, the p-component of the dimension of the irreducible module divides the order of the group; this was Problem 17 in a list of open problems compiled by R. Brauer in 1963 (cf. [9], [21], p. 166). Even this conjecture was refuted when J. G. Thackray showed in 1981 that McLaughlin's simple group of order $2^7 \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11$ has a simple module of dimension $2^9 \cdot 7$ in characteristic 2 (cf. [98]). Of course, the result holds for group rings over algebraically closed fields of characteristic zero, or if the characteristic of the field does not divide the order of the group. More generally, it holds for p-solvable groups (cf. [15], Cor. (22.5), p. 518).

The first result on the conjecture for arbitrary Hopf algebras is due to R. G. Larson. This result also shows why the problem is more difficult for Hopf algebras than for groups. As noted above, the conjecture is true for group rings of finite groups over algebraically closed fields of characteristic zero (cf. [15], Prop. (9.32), p. 216). The proof of this result exploits properties of algebraic integers, and, in doing so, exploits the fact that the group ring $K[G]$ has an obvious integral form, namely $\mathbb{Z}[G]$. Larson's result says that the same holds if the Hopf algebra under consideration is defined over $\mathbb{Z}$. More precisely, Larson proves (cf. [33], Prop. 4.2, p. 208):

Theorem Suppose that R is a Dedekind domain with fraction field K and that H is a Hopf algebra over R that is finitely generated projective over R. Assume that $H \otimes_R K$ is split semisimple and involutory. Denote the set of left integrals in H by L. Then, for every simple $H \otimes_R K$-module V, the principal ideal $(\dim V)$ of R divides the ideal $\epsilon_H(L)$, i. e., we have $(\dim V) \supseteq \epsilon_H(L)$.

Variants of this theorem can also be found in [1], Thm. (4.6), p. 3252, [75], and the earlier preprint versions of [20].

6.2 An important result on Kaplansky's sixth conjecture is a theorem of W. D. Nichols and M. B. Richmond that answers the question affirmatively if the simple module is two-dimensional (cf. [61], Cor. 12, p. 306):

Theorem The dimension of a semisimple Hopf algebra over an algebraically closed field is even if the Hopf algebra has a simple module of dimension 2.

The technique of the authors is to analyse the possible decompositions of the tensor product of the two-dimensional simple module and its dual. More precisely, the argument proceeds in the way that either this decomposition behaves similar to the decomposition of the regular module of the Lie algebra $sl(2)$, in which case the Hopf algebra considered is infinite-dimensional, or the corresponding Hopf algebra admits certain Hopf algebra quotients of dimension 2, 12, 24, or 60 whose representations are similar to those of the dihedral groups D_n, the tetrahedral group A_4, the octahedral (resp. hexahedral) group S_4, or the icosahedral (resp. dodecahedral) group A_5. The authors have continued their investigation in [62].

6.3 Kaplansky's sixth conjecture has also been considered by S. Montgomery and S. J. Witherspoon (cf. [52]). They prove that, in certain situations, Kaplansky's conjecture holds for a Hopf algebra if it holds for a subalgebra. More precisely, they prove (cf. [52], Cor. 1.4, p. 319, Thm. 2.1, p. 320):

Theorem Suppose that A is a finite-dimensional algebra over an algebraically closed field such that the dimensions of the simple modules of A divide the dimension of A. Suppose that G is a finite group such that the characteristic of the base field does not divide the order of G. If B is a crossed product of A and the group ring $K[G]$, then the dimensions of the simple modules of B divide the dimension of B. This also holds if B is a crossed product of A and the dual group ring $K[G]^*$.

Using results of G. I. Kac and A. Masuoka (cf. [27], [46]), Montgomery and Witherspoon then show that Hopf algebras of prime power dimension over algebraically closed fields of characteristic zero can be constructed by iterating the crossed product constructions described in the theorem. Therefore, Kaplansky's sixth conjecture holds for these Hopf algebras (cf. [52], Cor. 3.6, p. 325).

6.4 Kaplansky's sixth conjecture has also been verified for other classes of Hopf algebras. First, Y. Zhu proved that, for a semisimple Hopf algebra H over an algebraically closed field of characteristic zero, the dimensions of the simple $D(H)$-submodules of H divide the dimension of H (cf. [105], p. 2850), where $D(H)$ denotes the Drinfel'd double of H. His argument relies, besides the so-called class equation that will be discussed below, on the fact that the action of the Drinfel'd double on H precisely centralizes the action of the character algebra on H. Zhu's result was then generalized by P. Etingof and S. Gelaki to arbitrary simple modules of the Drinfel'd double $D(H)$. As a consequence, they were able to obtain the following theorem (cf. [18], Thm. 1.5, p. 194, [20], Cor. 3.8, p. 860):

Theorem Suppose that H is a semisimple, cosemisimple, quasitriangular Hopf algebra over an algebraically closed field. Then the dimension of every simple H-module divides the dimension of H.

Their proof has already been simplified by Y. Tsang and Y. Zhu (cf. [99]), and by H.-J. Schneider (cf. [83]).

7 The eighth conjecture

7.1 Kaplansky's eighth conjecture states that a Hopf algebra of prime dimension over an algebraically closed field is commutative and cocommutative. Since commutativity properties remain unchanged under extensions of the base field, the assumption of algebraic closure is again not relevant here. However, in the algebraically closed case, something more can be said. As observed by P. Cartier and D. K. Harrison (cf. [14], p. 102, [30], Thm. 3.2, p. 354, [50], Thm. 2.3.1, p. 22), finite-dimensional, cocommutative, cosemisimple Hopf algebras over algebraically closed fields are group rings. As a consequence of the Nichols-Zoeller theorem, it was shown by R. G. Larson and D. E. Radford that a Hopf algebra of prime dimension is semisimple and cosemisimple if the characteristic of the base field is zero or greater than the dimension of the Hopf algebra (cf. [37], Thm. 2.3, p. 12). Therefore, the conjecture would imply in these cases that the only example of a Hopf algebra of prime dimension is the group ring of the cyclic group of prime order. Of course, this does not hold if the characteristic of the base field coincides with the dimension of the Hopf algebra, because then the restricted enveloping algebra of the base field, considered an abelian restricted Lie algebra via the Frobenius homomorphism, is another example of a commutative and cocommutative Hopf algebra of dimension p.

The history of Kaplansky's eighth conjecture is slightly strange. Kaplansky himself had considered the dimensions 2, 3, and 5 in the first appendix of his notes. He added as a remark to the conjecture that G. I. Kac had partial results on this conjecture, and his bibliography also contains the corresponding reference [27]. The conjecture was then considered as open for about twenty years; the only contribution was given by C. R. Cai and H. X. Chen, who confirmed the conjecture for dimension 7 and 13 over fields of characteristic zero (cf. [10]). A year later, Y. Zhu was able to prove the conjecture completely over fields of characteristic zero, using an argument based on the Drinfel'd double construction (cf. [103]). According to Zhu, E. Effros then pointed out Kac's work to him, and upon reading it he discovered that Kac' work already contained generalizations of his techniques. Zhu then decided to follow essentially Kac' argument in the published version of his manuscript (cf. [104]), which made it possible to avoid the Drinfel'd double construction.

The argument of Kac and Zhu is based on a theorem which is now called the class equation, and may be considered as even more important than the solution of the conjecture itself:

Theorem Suppose that H is a semisimple Hopf algebra over an algebraically closed field of characteristic zero. Suppose that $e_1, \ldots, e_m$ is a complete set of primitive orthogonal idempotents of the character ring $\mathrm{Ch}(H)$, where e_1 is an integral of H^*. Then we have

$$\dim H = \sum_{i=1}^{m} \dim e_i H^*$$

$\dim e_1 H^* = 1$, and $\dim e_i H^*$ divides $\dim H$.

Note that the nontrivial part is the divisibility statement. The theorem reduces to the ordinary class equation if applied to the group ring of a finite group, because the idempotents above then become the characteristic functions of the conjugacy classes. If applied to the dual group ring, the theorem implies that the dimensions of the simple modules divide the order of the group. A simplified proof of the class equation has been given by M. Lorenz (cf. [39]). S. J. Witherspoon has related the class equation to analogues of conjugacy class sums (cf. [101]).

As a consequence of their lifting theorem already discussed in conjunction with the fifth conjecture, P. Etingof and S. Gelaki proved, over any field, that a semisimple, cosemisimple Hopf algebra of prime dimension is commutative and cocommutative (cf. [20], Thm. 3.4, p. 859). They then reprove the above-mentioned result of Larson and Radford to find conditions when these semisimplicity assumptions are satisfied (cf. [20], Cor. 3.4, p. 859).

7.2 Nowadays, Kaplansky's eighth conjecture may be understood as the first step in the program to understand the structure of semisimple Hopf algebras in terms of the prime factors of their dimension, similar to the situation in finite group theory. This program, also called the classification program for semisimple Hopf algebras, has recently experienced considerable progress, the class equation being an important instrument in this investigation. One of the first consequences of the class equation is the observation that, over algebraically closed fields of characteristic zero, a semisimple Hopf algebra of prime power dimension contains a nontrivial central grouplike element (cf. [27], [46]). Here, the proof is analogous to the proof that a p-group has nontrivial center. As above, the lifting theorem of Etingof and Gelaki can be used to generalize this result to semisimple, cosemisimple Hopf algebras over fields of positive characteristic (cf. [20], Thm. 3.10, p. 861). As a consequence, A. Masuoka has proved that, over a field of characteristic zero, semisimple Hopf algebras of dimension p^2 are commutative and cocommutative. A. Masuoka has also contributed several other results (cf. [44], [45], [47], [48], [49]). Further results can be found in [23], [19], [86], and [53]. Since this development is reviewed in [51], we shall not give further details here.

8 The tenth conjecture

8.1 Kaplansky's tenth conjecture states that, over an algebraically closed field, the number of isomorphism classes of Hopf algebras of a given dimension is finite, provided that the characteristic of the base field does not divide this dimension. This was proved by D. Ştefan for isomorphism classes of semisimple and cosemisimple Hopf algebras (cf. [87]). Note that, due to the positive solution of the seventh conjecture, for a semisimple and cosemisimple Hopf algebra the characteristic does not divide the dimension. Ştefan's proof is based on homological techniques as well as techniques from algebraic geometry. A simplified proof, which is, however, based on similar techniques, has been given by H.-J. Schneider (cf. [82]). A relative version of this theorem was given by P. Etingof and S. Gelaki ([20], Thm. 1.2, p. 852).

8.2 However, the conjecture is false in general. This was realized more or less simultaneously by three groups of researchers in 1997, namely N. Andruskiewitsch and H.-J. Schneider on the one hand, M. Beattie, S. Dăscălescu, and L. Grünenfelder on the other hand, and also by S. Gelaki (cf. [3], [6], [22]). In the approach of the first two groups, the conjecture is refuted by constructing pointed Hopf algebras of dimension p^4 for an odd prime p, whereas in Gelaki's approach infinite families of Hopf algebras of dimension mn^2 are constructed, where $n > 2$ and $m > n$ are natural numbers such that n divides m. Such a construction, from which the Hopf algebras of dimension p^4 are obtained as a special case, can also be found in the article by Andruskiewitsch and Schneider; their method also leads to the classification of pointed Hopf algebras of dimension p^3 over an algebraically closed field of characteristic zero, where p is an odd prime. This classification has also be obtained by S. Caenepeel and S. Dăscălescu as well as by D. Ştefan and F. van Oystaeyen (cf. [11], [88]).

The approach of Andruskiewitsch and Schneider is part of a larger program to investigate the structure of pointed Hopf algebras. Their program can be described as follows: A pointed Hopf algebra comes with a natural filtration, the so-called coradical filtration. The associated graded Hopf algebra can be decomposed into a Radford biproduct of the coradical of the original Hopf algebra, which is a group ring by assumption, and a Yetter-Drinfel'd Hopf algebra over that group ring. The Yetter-Drinfel'd Hopf algebra is often generated by primitive elements. The program now is to determine these Yetter-Drinfel'd Hopf algebras, and therefore also the corresponding Radford biproducts, and then to determine all possible liftings of this graded Hopf algebra to a filtered Hopf algebra. This program has recently undergone considerable progress (cf. [4]), leading to a complete classification of nonsemisimple pointed Hopf algebras of dimension p^4 over an algebraically closed field of characteristic zero, where p is an odd prime (cf. [5], see [13] for the even case). Note that, in this situation, a pointed semisimple Hopf algebra is a group ring. M. Graña has carried out part of this program even of pointed Hopf algebras of dimension p^5 (cf. [24], [25]). Also the other groups mentioned above have continued their investigations (cf. [7], [8], [12]).

9 Appendix

In this appendix, we reproduce literally appendix 2 from [28], which contains Kaplansky's conjectures. It may be helpful for the understanding of the conjectures to know that Kaplansky's lecture notes contain a bibliography which, among many other references, lists the items labelled [27], [32], and [64] in the present article. Kaplansky's conjectures are:

1. If C is a Hopf subalgebra of the Hopf algebra B then B is a free left C-module. (Remark. Nichols has proved this if B contains the coradical of C.)

2. Call a coalgebra C admissible if it admits an algebra structure making it a Hopf algebra. The conjecture states that C is admissible if and only if every finite subset of C lies in a finite-dimensional admissible subcoalgebra. (Remarks. 1. Both implications seem hard. 2. There is a corresponding conjecture where "Hopf algebra" is replaced by "bialgebra". 3. There is a dual conjecture for locally finite algebras.)

3. A Hopf algebra of characteristic 0 has no non-zero central nilpotent elements.

4. (Nichols). Let x be an element in a Hopf algebra H with antipode S. Assume that for any a in H we have
$$\sum b_i x S(c_i) = \epsilon(a)x$$
where $\Delta a = \sum b_i \otimes c_i$. Conjecture: x is in the center of H.

In the remaining six conjectures H is a finite-dimensional Hopf algebra over an algebraically closed field.

5. If H is semisimple on either side (i.e. either H or the dual H^* is semisimple as an algebra) the square of the antipode is the identity.

6. The size of the matrices occurring in any full matrix constituent of H divides the dimension of H.

7. If H is semisimple on both sides the characteristic does not divide the dimension.

8. If the dimension of H is prime then H is commutative and cocommutative.

Remark. Kac, Larson, and Sweedler have partial results on 5 - 8.

In the two final conjectures assume that the characteristic does not divide the dimension of H.

9. The dimension of the radical is the same on both sides.

10. There are only a finite number (up to isomorphism) of Hopf algebras of a given dimension.

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