Kac-Moody algebras

Yorck Sommerhäuser

Presented at the Ring Theory Conference, Miskolc, Hungary, July 21-26, 1996

Abstract

We construct symmetrizable Kac-Moody algebras via a generalized semidirect product construction.

1 Introduction

Kac-Moody algebras are a special kind of Lie algebras that generalize the class of finite dimensional semisimple Lie algebras. They were introduced by V. G. Kac and R. V. Moody in 1967 (cf. [8], [10]). They are defined via generators and relations and some additional construction. It was shown much later by O. Gabber and V. G. Kac that Kac-Moody algebras arising from symmetrizable generalized Cartan matrices can be defined entirely via generators and relations (cf. [6]).

In the present article, we give an alternative construction of symmetrizable Kac-Moody algebras which is not based on generators and relations at all. The starting point of the present construction is the so-called triangular decomposition, which describes Kac-Moody algebras as the vector space direct sum of three Lie subalgebras which should be viewed as generalizations of the strictly upper triangular matrices, the diagonal matrices and the strictly lower triangular matrices respectively. Here, we derive a general construction for gluing three Lie algebras to a single one in such a way that the original three Lie algebras appear as subalgebras of the new one, and apply this construction to symmetrizable Kac-Moody algebras. The spirit of the construction is very close to the well-known semidirect product of Lie algebras. It is an analogue of a similar construction for the deformed enveloping algebras of Kac-Moody algebras (cf. [15], [16]).

The article is organized as follows: In the preliminary Section 2, we review some background material and introduce some basic notions that will be used in the sequel. In particular, we introduce the notion of a Yetter-Drinfel'd Lie algebra. This kind of Lie algebras will be the main object in Section 4. The notion of a Yetter-Drinfel'd Lie algebra introduced here is not interrelated with the notion of a Lie algebra in the category of Yetter-Drinfel'd modules considered in [12], [13]. In Section 3, we carry out a general framework construction which determines completely the Lie algebras which can be decomposed into three Lie subalgebras in a certain way. Here, we proceed in a completely analogous fashion as in the first construction in [15], and at the end of the Section we briefly explain the connection of the two constructions. We also look at a special case of the first construction for Lie coalgebras, a construction that we call the two-sided cosemidirect product, and explain how the so-called bismash product of Lie algebras arises as a special case of the first construction. In Section 4, we discuss a special case of this construction which can be used if two of the three Lie algebras are Yetter-Drinfel'd Lie algebras that are dual in a certain sense. Although conceptually very similar, this construction can not be considered as a special case of the second construction in [15], because the applicability of this construction would require that the enveloping algebras of the two dual Lie algebras are dual too, which never happens unless both Lie algebras are abelian. We discuss further special cases of this second construction, which arise if the Yetter-Drinfel'd Lie algebras involved are symmetric or skew-symmetric, as well as the cases where the background Lie bialgebra is quasitriangular or coquasitriangular. We also explain how the classical double construction of V. G. Drinfel'd arises as a special case of the second construction. In the last Section, we describe a cohomological method to construct a bilinear form between two free Lie algebras. We then show that by factoring out the radical of the bilinear form and then applying the second construction one can construct symmetrizable Kac-Moody algebras.

All vector spaces occurring in this article are defined over a base field that is denoted by K, except for the last Section, where we are working over the complex numbers as in [9]. The term 'linear map' will always mean 'K-linear map'.

2 Yetter-Drinfel'd Lie algebras

2.1 Recall that a Lie coalgebra is a vector space L together with a mapping $\Delta : L \rightarrow L \otimes L$, called the coproduct, that has the property that the induced product $L^* \otimes L^* \rightarrow L^*$ on the dual vector space turns the dual vector space into a Lie algebra (cf. [5], [2]). We shall use the following variant of the Heyneman-Sweedler sigma notation (cf. [17]): We write the coproducts using lower indices:

$$\Delta(y) = y_1 \otimes y_2$$

for any $y \in L$. This enables us to write tensors as if they were decomposable. Since we do not consider coproducts that are coassociative, the repeated application of the coproduct will for example be denoted as:

$$(\Delta \otimes id_L)\Delta(y) = y_{11} \otimes y_{12} \otimes y_2$$

Definition

1. If L is a Lie coalgebra and V is a vector space, a linear map $\delta : V \rightarrow L \otimes V$ is called a positive coaction if it satisfies:

$$(\Delta \otimes id_V) \circ \delta = (id_L \otimes \delta) \circ \delta - (\tau \otimes id_V) \circ (id_L \otimes \delta) \circ \delta$$

Here $\tau : L \otimes L \to L \otimes L$ denotes the map that interchanges the tensorands. V together with a positive coaction will be called a positive Lie comodule.

2. If L is a Lie coalgebra and V is a vector space, a linear map $\delta : V \to L \otimes V$ is called a negative coaction if it satisfies:

$$(\Delta \otimes id_V) \circ \delta = (\tau \otimes id_V) \circ (id_L \otimes \delta) \circ \delta - (id_L \otimes \delta) \circ \delta$$

V together with a negative coaction will be called a negative Lie comodule.

3. Vice versa, if L is a Lie coalgebra and V is a vector space, a linear map $\delta : L \to V \otimes L$ is called a coderivation if it satisfies:

$$(id_V \otimes \Delta) \circ \delta = (\delta \otimes id_L) \circ \Delta + (\tau \otimes id_L) \circ (id_L \otimes \delta) \circ \Delta$$

For coactions, coderivations, and similar linear maps we shall use the Heyneman-Sweedler notation with upper indices:

$$\delta(v) = v^1 \otimes v^2$$

For example, the defining condition for a positive coaction then reads:

$$v^1{}_1 \otimes v^1{}_2 \otimes v^2 = v^1 \otimes v^{21} \otimes v^{22} - v^{21} \otimes v^1 \otimes v^{22}$$

Of course, positive and negative coactions are not really different, since $-\delta$ is a negative coaction if δ is a positive coaction. Observe that positive and negative Lie comodules are equal if the Lie coalgebra structure on L is trivial, that is, the coproduct is equal to zero. Lie coalgebras whose coproduct is identically zero will also be called coabelian. The name 'coaction' resp. 'coderivation' is explained by the following Proposition:

Proposition

1. If L is a Lie coalgebra and $\delta : V \to L \otimes V$ is a positive coaction, then the mapping
$$\phi : L^* \to End(V^*), x^* \mapsto \phi(x^*)$$
with $(\phi(x^*)(v^*))(v) = x^*(v^1)v^*(v^2)$ for $x^* \in L^*, v^* \in V^*$ and $v \in V$ is a Lie algebra homomorphism.

2. If L is a Lie coalgebra and $\delta : L \to V \otimes L$ is a coderivation, then for all $v^* \in V^*$ the mapping
$$L^* \to L^*, x^* \mapsto (x \mapsto v^*(x^1)x^*(x^2))$$
is a derivation.

The easy proof is omitted.

2.2 Recall that a Lie bialgebra is a Lie algebra that is simultaneously a Lie coalgebra in such a way that the comultiplication is a 1-cocycle (cf. [5], [2], for the definition of 1-cocycles cf. also Subsection 5.1).

Definition Suppose that L is a Lie bialgebra and that V is an L-module via $\phi : L \to End(V)$.

1. V together with a positive coaction $\delta : V \to L \otimes V$ is called a positive Yetter-Drinfel'd Lie module if the action and the coaction are compatible in the following way: For all $h \in L$ and all $v \in V$, we have:

$$\delta(\phi(h)(v)) = [h, v^1] \otimes v^2 + v^1 \otimes \phi(h)(v^2) + h_1 \otimes \phi(h_2)(v)$$

2. V together with a negative coaction $\delta : V \to L \otimes V$ is called a negative Yetter-Drinfel'd Lie module if the action and the coaction are compatible in the following way: For all $h \in L$ and all $v \in V$, we have:

$$\delta(\phi(h)(v)) = [h, v^1] \otimes v^2 + v^1 \otimes \phi(h)(v^2) - h_1 \otimes \phi(h_2)(v)$$

It is obvious that a positive Yetter-Drinfel'd Lie module becomes a negative Yetter-Drinfel'd Lie module if the coaction δ is replaced by $-\delta$. A Yetter-Drinfel'd Lie module V will be called symmetric (resp. skew-symmetric) if in addition the following condition is satisfied: For all $v, v' \in V$ we have:

$$\phi(v'^1)(v) \otimes v'^2 = v^2 \otimes \phi(v^1)(v')$$

(resp. $\phi(v'^1)(v) \otimes v'^2 = -v^2 \otimes \phi(v^1)(v')$.) These conditions should be compared with the main assumption in [15].

2.3 The notion of a Yetter-Drinfel'd Lie module is self-dual. This statement can be formalized by saying that the dual vector space of a Yetter-Drinfel'd Lie module is again a Yetter-Drinfel'd Lie module. We shall formalize the relation of a Yetter-Drinfel'd Lie module and its dual vector space in the following definition:

Definition Suppose that V and W are Yetter-Drinfel'd Lie modules over the Lie bialgebra L, either both positive or both negative. A bilinear form

$$\langle \cdot, \cdot \rangle : V \times W \to K$$

is called a Lie biform if the following equations hold:

1. $v^1 \langle v^2, w \rangle = -w^1 \langle v, w^2 \rangle$
2. $\langle \phi_V(h)(v), w \rangle = -\langle v, \phi_W(h)(w) \rangle$

for all $v \in V, w \in W$ and $h \in L$.

The interrelation between a Yetter-Drinfel'd Lie module and its dual vector space is now the following one:

Proposition Suppose that V is a finite dimensional positive (resp. negative) Yetter-Drinfel'd Lie module. Then the dual vector space V^* has a unique structure as a positive (resp. negative) Yetter-Drinfel'd Lie module such that the evaluation

$$\langle \cdot, \cdot \rangle : V^* \times V \to K, (f, v) \mapsto f(v)$$

is a Lie biform.

The proof is based on direct verification and is omitted.

2.4 Recall that a Lie bialgebra is called quasitriangular if its coproduct comes from a so-called classical r-matrix. That is, there is an element $r = \sum_{i=1}^{n} a_i \otimes b_i$ that satisfies the so-called classical Yang-Baxter equation

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0$$

such that the coproduct is the coboundary of r:

$$\Delta(h) = h.r = \sum_{i=1}^{n} ([h, a_i] \otimes b_i + a_i \otimes [h, b_i])$$

Lie bialgebras whose coproduct is given by the last formula for some r that does not necessarily satisfy the classical Yang-Baxter equation are called coboundary Lie bialgebras. Quasitriangular Lie bialgebras whose r-matrix is skew-symmetric are also called triangular. (Cf. [5] and [2], p. 54 for more details and the explanation of the notation.)

In the case of quasitriangular Lie bialgebras, every module can be turned into a Yetter-Drinfel'd Lie module in the following way:

Proposition Suppose that L is a quasitriangular Lie bialgebra with classical r-matrix $r = \sum_{i=1}^{n} a_i \otimes b_i$, and suppose that $\phi : L \to End(V)$ is a representation.

1. V becomes a positive Yetter-Drinfel'd Lie module via the coaction

$$\delta(v) = \sum_{i=1}^{n} b_i \otimes \phi(a_i)(v)$$

2. V becomes a negative Yetter-Drinfel'd Lie module via the coaction

$$\delta(v) = \sum_{i=1}^{n} a_i \otimes \phi(b_i)(v)$$

Proof. We only prove the first assertion, the proof of the second one is rather similar. The fact that δ is a positive coaction follows directly from the classical Yang-Baxter equation. The compatibility condition between action and coaction also holds for coboundary Lie bialgebras:

$$\begin{aligned}
[h, v^1] \otimes v^2 + v^1 \otimes \phi(h)(v^2) - h_2 \otimes \phi(h_1)(v) \\
&= \sum_i [h, b_i] \otimes \phi(a_i)(v) + \sum_i b_i \otimes \phi(h)(\phi(a_i)(v)) \\
&\quad - \sum_i [h, b_i] \otimes \phi(a_i)(v) - \sum_i b_i \otimes \phi([h, a_i])(v) \\
&= \sum_i b_i \otimes \phi(a_i)(\phi(h)(v)) = \delta(\phi(h)(v))
\end{aligned}$$

□

It is obvious that V becomes a symmetric (resp. skew-symmetric) Yetter-Drinfel'd Lie module if the r-matrix is symmetric (resp. skew-symmetric).

2.5 Dual to the notion of a quasitriangular Lie bialgebra, there is the notion of a coquasitriangular Lie bialgebra. Since this notion is not so ubiquitous in the literature, we recall the definition:

Definition A Lie bialgebra is called coquasitriangular if there is a bilinear form

$$\langle \cdot, \cdot \rangle_0 : L \times L \to K,$$

called the dual r-matrix, that satisfies

$$\forall h, h' \in L : [h, h'] = h_1 \langle h_2, h' \rangle_0 + h'_1 \langle h, h'_2 \rangle_0$$

and in addition one of the following equivalent conditions holds:

1. (Dual Yang-Baxter equation:) For all $h, h', h'' \in L$ we have:

$$\langle h_1, h' \rangle_0 \langle h_2, h'' \rangle_0 + \langle h, h'_1 \rangle_0 \langle h'_2, h'' \rangle_0 + \langle h, h''_1 \rangle_0 \langle h', h''_2 \rangle_0 = 0$$

2. For all $h, h', h'' \in L$ we have:

$$\langle [h, h'], h'' \rangle_0 + \langle h', [h, h''] \rangle_0 = \langle h', h_1 \rangle_0 \langle h_2, h'' \rangle_0$$

3. For all $h, h', h'' \in L$ we have:

$$\langle h \otimes h', \Delta(h'') \rangle_0 = \langle [h, h'], h'' \rangle_0$$

4. For all $h, h', h'' \in L$ we have:

$$\langle \Delta(h), h' \otimes h'' \rangle_0 = -\langle h, [h', h''] \rangle_0$$

This definition is set up in such a way that in the case of finite dimensional Lie bialgebras, L is coquasitriangular if and only if the dual vector space L^* is a quasitriangular Lie bialgebra. A coquasitriangular Lie bialgebra whose dual r-matrix is skew-symmetric is called cotriangular. Since we see from the last two formulations of the dual Yang-Baxter equation that the dual r-matrix relates the Lie algebra structures on L and L^*, the following Proposition comes at no surprise:

Proposition 1 If L is a finite dimensional coquasitriangular Lie bialgebra with a nondegenerate dual r-matrix $\langle \cdot, \cdot \rangle_0$, then L is also quasitriangular with r-matrix $r = \sum_{i=1}^n a_i \otimes b_j$, where $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ are bases of L that satisfy:

$$\langle b_i, a_j \rangle_0 = \delta_{ij}$$

It should be observed that, as in the quasitriangular case, there are always at least two possible choices of the dual r-matrix: If $\langle \cdot, \cdot \rangle_0$ is a dual r-matrix, then the bilinear form

$$L \times L \to K, (h, h') \mapsto -\langle h', h \rangle_0$$

is another choice of a dual r-matrix.

As in the quasitriangular case, coquasitriangular structures can be used to turn Lie comodules into Yetter-Drinfel'd Lie modules:

Proposition 2 Suppose that L is a coquasitriangular Lie bialgebra with dual r-matrix $\langle \cdot, \cdot \rangle_0$.

1. If V is a positive Lie comodule, then V becomes a positive Yetter-Drinfel'd Lie module by introducing the action

$$\phi : L \to End(V), h \mapsto (v \mapsto \langle v^1, h \rangle_0 v^2)$$

2. If V is a negative Lie comodule, then V becomes a negative Yetter-Drinfel'd Lie module by introducing the action

$$\phi : L \to End(V), h \mapsto (v \mapsto \langle h, v^1 \rangle_0 v^2)$$

The proof is based on direct verification and is omitted. It it obvious that the above Yetter-Drinfel'd Lie module structures are symmetric (resp. skew-symmetric) if the dual r-matrix is a symmetric (resp. skew-symmetric) bilinear form.

2.6 For every Yetter-Drinfel'd Lie module, there is an interesting equivariant map from the second exterior power to the second tensor power of the module. Here and in the sequel, we shall use the term 'equivariant map' instead of 'L-module homomorphism'. Since not all of the Yetter-Drinfel'd compatibility conditions are really needed, we state a slightly more general statement.

Proposition Suppose that L is a Lie algebra and that $\phi : L \to End(V)$ is a representation. Suppose that we are given linear mappings

$$\Delta : L \to L \otimes L \qquad \delta : V \to L \otimes V$$

that are compatible in the following way:

$$\forall h \in L\, \forall v \in V : \delta(\phi(h)(v)) = [h, v^1] \otimes v^2 + v^1 \otimes \phi(h)(v^2) + h_1 \otimes \phi(h_2)(v)$$

Then the mapping

$$f : V \wedge V \to V \otimes V$$
$$v \wedge v' \mapsto v^2 \otimes \phi(v^1)(v') - \phi(v^1)(v') \otimes v^2 + \phi(v'^1)(v) \otimes v'^2 - v'^2 \otimes \phi(v'^1)(v)$$

is equivariant, where tensor and exterior powers are L-modules via the product rule.

Proof. We first consider the mapping

$$g : V \otimes V \to V \otimes V, v \otimes v' \mapsto v^2 \otimes \phi(v^1)(v') + \phi(v'^1)(v) \otimes v'^2$$

We now calculate:

$$\begin{aligned}
g(\phi(h).(v \otimes v')) &= g(\phi(h)(v) \otimes v' + v \otimes \phi(h)(v')) \\
&= v^2 \otimes \phi([h, v^1])(v') + \phi(h)(v^2) \otimes \phi(v^1)(v') \\
&\quad + \phi(h_2)(v) \otimes \phi(h_1)(v') + \phi(v'^1)(\phi(h)(v)) \otimes v'^2 \\
&\quad + v^2 \otimes \phi(v^1)(\phi(h)(v')) + \phi([h, v'^1])(v) \otimes v'^2 \\
&\quad + \phi(v'^1)(v) \otimes \phi(h)(v'^2) + \phi(h_1)(v) \otimes \phi(h_2)(v') \\
&= v^2 \otimes \phi(h)(\phi(v^1)(v')) + \phi(h)(v^2) \otimes \phi(v^1)(v') \\
&\quad + \phi(h)(\phi(v'^1)(v)) \otimes v'^2 + \phi(v'^1)(v) \otimes \phi(h)(v'^2)
\end{aligned}$$

Since we have $f(v \wedge v') = g(v \otimes v' - v' \otimes v)$, the assertion follows. □

2.7 Suppose that L_0 is a Lie algebra. The notion of a Yetter-Drinfel'd Lie algebra that we introduce now will become important in the second construction.

Definition A Yetter-Drinfel'd Lie algebra over L_0 is a Lie algebra L which is simultaneously a Lie coalgebra with coproduct

$$\Delta : L \to L \otimes L$$

for which we are given a coderivation $\delta : L \to L_0 \otimes L$ and a Lie algebra homomorphism $\phi : L_0 \to Der(L)$ such that the following compatibility conditions are satisfied:

1. δ is a 1-cocycle. That is, for all $y, y' \in L$ we have:

$$\delta([y, y']) = y^1 \otimes [y^2, y'] + y'^1 \otimes [y, y'^2]$$

   Here we have viewed L_0 as a trivial L-module.

2. Δ is equivariant. That is, for all $h \in L_0, y \in L$ we have:

$$\Delta(\phi(h)(y)) = \phi(h).\Delta(y)$$

   Here $\phi(h)$ acts on the tensor product via the product rule.

3. The coboundary of Δ is determined by ϕ and δ. That is, for all $y, y' \in L$ we have:

$$\begin{aligned}&\Delta([y, y']) - y.\Delta(y') + y'.\Delta(y) \\ &\quad = y^2 \otimes \phi(y^1)(y') - \phi(y^1)(y') \otimes y^2 + \phi(y'^1)(y) \otimes y'^2 - y'^2 \otimes \phi(y'^1)(y)\end{aligned}$$

A Yetter-Drinfel'd Lie algebra will be called positive if L_0 is also a Lie bialgebra and δ turns L into a positive Yetter-Drinfel'd Lie module, similarly it will be called negative if δ turns L into a negative Yetter-Drinfel'd Lie module. Note that a positive Yetter-Drinfel'd Lie algebra becomes negative if δ is replaced by $-\delta$ and Δ is replaced by $-\Delta$. Condition 3 above simplifies if L is a symmetric or skew-symmetric Yetter-Drinfel'd Lie module: In the symmetric case, it reads:

$$\Delta([y, y']) - y.\Delta(y') + y'.\Delta(y) = 2\ y^2 \otimes \phi(y^1)(y') - 2\ \phi(y^1)(y') \otimes y^2$$

In the skew-symmetric case, it reads:

$$\Delta([y, y']) - y.\Delta(y') + y'.\Delta(y) = 0$$

so that skew-symmetric Yetter-Drinfel'd Lie algebras are in particular Lie bialgebras. We finally note that the notion of a Yetter-Drinfel'd Lie algebra used here is not related to the notion of a Lie algebra inside the category of Yetter-Drinfel'd modules as in [12], [13].

2.8 The dual vector space of a finite dimensional positive Yetter-Drinfel'd Lie algebra is again a positive Yetter-Drinfel'd Lie algebra. Since in our main application in the last Section we shall consider infinite dimensional Yetter-Drinfel'd Lie algebras, we formalize the relationship between a Yetter-Drinfel'd Lie algebra and its dual vector space by introducing the notion of a Lie algebra biform.

Definition Suppose that L_- and L_+ are Yetter-Drinfel'd Lie algebras over the Lie algebra L_0. A bilinear form

$$\langle \cdot, \cdot \rangle : L_- \times L_+ \to K, (x, y) \mapsto \langle x, y \rangle$$

is called a Lie algebra biform if the following conditions are satisfied:

1. For all $x\in L_-$, $h\in L_0$ and $y\in L_+$ we have: $\langle x, \phi_+(h)(y)\rangle = -\langle \phi_-(h)(x), y\rangle$
2. For all $x \in L_-$ and $y, y' \in L_+$ we have: $\langle x_1, y\rangle\langle x_2, y'\rangle = \langle x, [y, y']\rangle$
3. For all $x, x' \in L_-$ and $y \in L_+$ we have: $\langle x, y_1\rangle\langle x', y_2\rangle = \langle [x, x'], y\rangle$

Lie algebra biforms are not necessarily Lie biforms, since we have not assumed that the Lie algebra biform is compatible with the coaction. We shall call a Lie algebra biform strict if in addition the following condition is satisfied:

4. For all $x \in L_-$ and $y \in L_+$ we have: $x^1\langle x^2, y\rangle = -y^1\langle x, y^2\rangle$

The remark at the beginning of this Subsection can now be formalized as follows:

Proposition Suppose that L_+ is a finite dimensional positive Yetter-Drinfel'd Lie algebra over L_0. Then there is a unique positive Yetter-Drinfel'd Lie algebra structure on the dual vector space $L_- := L_+^*$ such that the evaluation

$$\langle \cdot, \cdot\rangle : L_- \times L_+ \to K, (f, y) \mapsto f(y)$$

becomes a strict Lie algebra biform.

Proof. The uniqueness is clear. For the existence, observe that the axioms in the definition of a strict Lie algebra biform obviously also can serve as a definition of the corresponding structures on the dual vector space L_-. It remains to show that these structures in fact satisfy the properties of a positive Yetter-Drinfel'd Lie algebra, which is a straightforward verification. $\Box$

2.9 The existence of a Lie algebra biform between two Yetter-Drinfel'd Lie algebras leads to a natural action of one Yetter-Drinfel'd Lie algebra on the other. This action is dual to the adjoint action with respect to the Lie algebra biform.

Definition Suppose that

$$\langle \cdot, \cdot\rangle : L_- \times L_+ \to K$$

is a Lie algebra biform between two Yetter-Drinfel'd Lie algebras L_- and L_+. We define the coadjoint action ψ_+ of L_- on L_+:

$$\psi_+ : L_- \to End(L_+)$$

by $\psi_+(x)(y) = y_1\langle x, y_2\rangle$ for all $x \in L_-, y \in L_+$. Similarly, we define the coadjoint action ψ_- of L_+ on L_-:

$$\psi_- : L_+ \to End(L_-)$$

by $\psi_-(y)(x) = x_1\langle x_2, y\rangle$ for all $x \in L_-, y \in L_+$.

Proposition 1 The coadjoint actions have the properties:

1. ψ_+ and ψ_- are Lie algebra homomorphisms.
2. For all $x, x' \in L_-$ and $y, y' \in L_+$ we have:

$$\langle x', \psi_+(x)(y)\rangle = -\langle [x, x'], y\rangle \qquad \langle \psi_-(y)(x), y'\rangle = -\langle x, [y, y']\rangle$$

In the case where the Lie algebras are finite dimensional and the bilinear form is nondegenerate, we can construct an invariant element from the bilinear form. This is a well-known procedure: The construction of the Casimir element from the Killing form rests on similar considerations. This technique is also used in the theory of Frobenius algebras.

Proposition 2 Suppose that

$$\langle \cdot, \cdot \rangle : L_- \times L_+ \to K$$

is a nondegenerate Lie algebra biform between two finite dimensional Yetter-Drinfel'd Lie algebras L_- and L_+. Select dual bases $x_1, \ldots, x_m \in L_-$ and $y_1, \ldots, y_m \in L_+$ that satisfy: $\langle x_i, y_j\rangle = \delta_{ij}$. Then we have:

1. For all $x \in L_-$ we have: $\sum_{i=1}^m ad(x)(x_j) \otimes y_j = -\sum_{i=1}^m x_j \otimes \psi_+(x)(y_j)$
2. For all $y \in L_+$ we have: $\sum_{i=1}^m \psi_-(y)(x_j) \otimes y_j = -\sum_{i=1}^m x_j \otimes ad(y)(y_j)$
3. For all $h \in L_0$, we have: $\sum_{i=1}^m \phi_-(h)(x_j) \otimes y_j = -\sum_{i=1}^m x_j \otimes \phi_+(h)(y_j)$

We omit the simple proof of this Proposition.

2.10 Of course, it may happen that a Lie algebra biform is degenerate. However, in the strict case there is always a nondegenerate quotient. In order to describe this quotient more conveniently, we introduce of morphisms between Yetter-Drinfel'd Lie algebras.

Definition Suppose that L_- and L_+ are Yetter-Drinfel'd Lie algebras over L_0. A Yetter-Drinfel'd Lie algebra map is an L_0-equivariant Lie algebra homomorphism $f : L_- \to L_+$ that satisfies in addition:

$$\Delta_+ \circ f = (f \otimes f) \circ \Delta_- \qquad \delta_+ \circ f = (id_{L_0} \otimes f) \circ \delta_-$$

Proposition Suppose that

$$\langle \cdot, \cdot \rangle : L_- \times L_+ \to K$$

is a strict Lie algebra biform between two Yetter-Drinfel'd Lie algebras L_- and L_+. Define the left and the right radical of the Lie algebra biform as:

$$R_- := \{x \in L_- | \forall y \in L_+ : \langle x, y\rangle = 0\} \qquad R_+ := \{y \in L_+ | \forall x \in L_- : \langle x, y\rangle = 0\}$$

Then there are unique Yetter-Drinfel'd Lie algebra structures on L_-/R_- resp. L_+/R_+ such that the canonical projections become Yetter-Drinfel'd Lie algebra maps. The induced bilinear form

$$L_-/R_- \times L_+/R_+ \to K, (\overline{x}, \overline{y}) \mapsto \langle x, y \rangle$$

is again a strict Lie algebra biform. If L_- and L_+ are both positive, then L_-/R_- and L_+/R_+ are also positive.

We leave the proof of the Proposition as an easy exercise.

2.11 As in the case of Yetter-Drinfel'd Lie modules, Yetter-Drinfel'd Lie algebras can be created from actions alone in case of coboundary Lie bialgebras:

Proposition Suppose that L_0 is a coboundary Lie bialgebra whose coproduct is the coboundary of $r = \sum_{i=1}^n a_i \otimes b_i$, and that $\phi : L_0 \to Der(L)$ is a Lie algebra homomorphism. Assume in addition that L is a Lie algebra and simultaneously a Lie coalgebra such that the following assumptions hold:

1. Δ_L is L_0-equivariant, that is, we have for all $h \in L_0, y \in L$:

$$\Delta_L(\phi(h)(y)) = \phi(h).\Delta_L(y)$$

2. The action determines the coboundary of the coproduct:

$$\begin{aligned}
&\Delta([y, y']) - y.\Delta(y') + y'.\Delta(y) \\
&= \sum_i \phi(a_i)(y) \otimes \phi(b_i)(y') - \sum_i \phi(a_i)(y') \otimes \phi(b_i)(y) \\
&\quad + \sum_i \phi(b_i)(y) \otimes \phi(a_i)(y') - \sum_i \phi(b_i)(y') \otimes \phi(a_i)(y)
\end{aligned}$$

Then we have two possibilities to turn L into a Yetter-Drinfel'd Lie algebra:

1. L becomes a Yetter-Drinfel'd Lie algebra via the coaction

$$\delta(y) = \sum_{i=1}^n b_i \otimes \phi(a_i)(y)$$

2. L becomes a Yetter-Drinfel'd Lie algebra via the coaction

$$\delta(y) = \sum_{i=1}^n a_i \otimes \phi(b_i)(y)$$

Proof. Note that the second statement follows from the first because the cooposite Lie bialgebra of L_0, which has the same Lie algebra structure but the negative coproduct, is also a coboundary Lie bialgebra with coproduct arising from $r^{cop} = \sum_{i=1}^n b_i \otimes a_i$. Therefore, we shall only prove the first assertion. It is surprising to see that this does require almost no properties of r. δ is a 1-cocycle because we have:

$$\begin{aligned}
\delta([y,y']) &= \sum_i b_i \otimes \phi(a_i)([y,y']) \\
&= \sum_i b_i \otimes [\phi(a_i)(y), y'] + \sum_i b_i \otimes [y, \phi(a_i)(y')] \\
&= y^1 \otimes [y^2, y'] + y'^1 \otimes [y, y'^2]
\end{aligned}$$

δ is a coderivation because we have:

$$\begin{aligned}
y^1 \otimes y^2{}_1 \otimes y^2{}_2 &= \sum_i b_i \otimes \Delta_L(\phi(a_i)(y)) \\
&= \sum_i b_i \otimes \phi(a_i)(y_1) \otimes y_2 + \sum_i b_i \otimes y_1 \otimes \phi(a_i)(y_2) \\
&= y_1{}^1 \otimes y_1{}^2 \otimes y_2 + y_2{}^1 \otimes y_1 \otimes y_2{}^2
\end{aligned}$$

$\square$

2.12 We have just seen that coactions can arise from actions. We shall now see that, vice versa, actions can arise from coactions if L_0 is coquasitriangular, more generally, if L_0 carries a bilinear form. In fact, we shall see at the end that both things are closely related.

Proposition 1 Suppose that L_0 is a Lie algebra and that

$$\langle \cdot, \cdot \rangle_0 : L_0 \times L_0 \to K$$

is a bilinear form. Suppose further that L is a Lie algebra which is simultaneously a Lie coalgebra. Let

$$\delta : L \to L_0 \otimes L$$

be some linear map, and define

$$\phi : L_0 \to End(L), h \mapsto (y \mapsto \langle y^1, h \rangle_0 y^2)$$

1. If δ is a 1-cocycle, that is, we have

$$\delta([y,y']) = y^1 \otimes [y^2, y'] + y'^1 \otimes [y, y'^2],$$

then $\phi(h)$ is a derivation.

2. Conversely, if $\langle \cdot, \cdot \rangle_0$ is nondegenerate and $\phi(h)$ is a derivation for all $h \in L_0$, then δ is a 1-cocycle.

3. If δ is a coderivation, then Δ is equivariant, that is, we have

$$\Delta(\phi(h)(y)) = \phi(h)(y_1) \otimes y_2 + y_1 \otimes \phi(h)(y_2)$$

4. Conversely, if $\langle\cdot,\cdot\rangle_0$ is nondegenerate and Δ is equivariant, then δ is a coderivation.

Proof. This is based on direct verification. $\Box$

The straightforward observations above imply the following result on Yetter-Drinfel'd Lie algebras:

Proposition 2 Suppose that L_0 is a coquasitriangular Lie bialgebra with dual r-matrix $\langle\cdot,\cdot\rangle_0$.

1. Suppose that L is a Lie algebra which is simultaneously a Lie coalgebra and a positive Lie comodule in such a way that the following properties hold:

   (a) δ is a 1-cocycle, that is, we have

   $$\delta([y,y']) = y^1 \otimes [y^2, y'] + y'^1 \otimes [y, y'^2].$$

   (b) δ is a coderivation.

   (c) We have for all $y, y' \in L$:

   $$\begin{aligned}\Delta([y,y']) &- y.\Delta(y') + y'.\Delta(y)\\ &= y^2 \otimes y'^2 \langle y'^1, y^1\rangle_0 - y'^2 \otimes y^2 \langle y'^1, y^1\rangle_0\\ &\quad + y^2 \otimes y'^2 \langle y^1, y'^1\rangle_0 - y'^2 \otimes y^2 \langle y^1, y'^1\rangle_0\end{aligned}$$

   Define

   $$\phi : L_0 \to End(L), h \mapsto (y \mapsto \langle y^1, h\rangle_0 y^2)$$

   Then L is a positive Yetter-Drinfel'd Lie algebra.

2. Suppose that L is a Lie algebra which is simultaneously a Lie coalgebra and a negative Lie comodule in such a way that the following properties hold:

   (a) δ is a 1-cocycle, that is, we have

   $$\delta([x,x']) = x^1 \otimes [x^2, x'] + x'^1 \otimes [x, x'^2].$$

   (b) δ is a coderivation.

   (c) We have for all $x, x' \in L$:

   $$\begin{aligned}\Delta([x,x']) &- x.\Delta(x') + x'.\Delta(x)\\ &= + x^2 \otimes x'^2 \langle x^1, x'^1\rangle_0 - x'^2 \otimes x^2 \langle x^1, x'^1\rangle_0\\ &\quad + x^2 \otimes x'^2 \langle x'^1, x^1\rangle_0 - x'^2 \otimes x^2 \langle x'^1, x^1\rangle_0\end{aligned}$$

Define

$$\phi : L_0 \to End(L), h \mapsto (x \mapsto \langle h, x^1\rangle_0 x^2)$$

Then L is a negative Yetter-Drinfel'd Lie algebra.

If L_0 is finite dimensional and the dual r-matrix $\langle\cdot,\cdot\rangle_0$ is nondegenerate, this can also be understood in terms of the preceding Subsection. For, in this case it is possible to select bases $a_1, \dots, a_n$ and $b_1, \dots, b_n$ of L_0 in such a way that:

$$\langle b_i, a_j\rangle_0 = \delta_{ij}$$

As explained in Subsection 2.5, L_0 can then also be regarded as quasitriangular with r-matrix $r = \sum_{i=1}^n a_i \otimes b_i$. But now, an action ϕ is induced by a coaction δ by the formula

$$\phi(h)(y) = \langle y^1, h\rangle_0 y^2$$

if and only if the coaction is induced by the action via

$$\delta(y) = \sum_{i=1}^n b_i \otimes \phi(a_i)(y)$$

3 The first construction

3.1 In this Section, we determine completely the general form of a Lie algebra L with the following property: L is the direct sum of three Lie subalgebras L_-, L_0 and L_+:

$$L = L_- \oplus L_0 \oplus L_+$$

such that $L_- \oplus L_0$ and $L_0 \oplus L_+$ are also Lie subalgebras of L containing L_- resp. L_+ as an ideal.

3.2 Suppose that L_-, L_0 and L_+ are three Lie algebras. Suppose further that there are four Lie algebra morphisms:

$$\begin{array}{ll} \phi_+ : L_0 \to Der(L_+) & \phi_- : L_0 \to Der(L_-) \\ \psi_+ : L_- \to End(L_+) & \psi_- : L_+ \to End(L_-) \end{array}$$

and a bilinear map:

$$\{\cdot,\cdot\} : L_- \times L_+ \to L_0, (x, y) \mapsto \{x, y\}$$

such that the following conditions are satisfied:

1. For all $x, x' \in L_-$ and for all $y \in L_+$ we have:

$$\begin{aligned} \psi_-(y)([x, x']) = & [\psi_-(y)(x), x'] + [x, \psi_-(y)(x')] \\ & + \phi_-(\{x, y\})(x') - \phi_-(\{x', y\})(x) \\ & + \psi_-(\psi_+(x')(y))(x) - \psi_-(\psi_+(x)(y))(x') \end{aligned}$$

For all $x \in L_-$ and for all $y, y' \in L_+$ we have:

$$\begin{aligned}\psi_+(x)([y,y']) = [\psi_+(x)(y), y'] + [y, \psi_+(x)(y')]\\ + \phi_+(\{x,y'\})(y) - \phi_-(\{x,y\})(y')\\ + \psi_+(\psi_-(y')(x))(y) - \psi_+(\psi_-(y)(x))(y')\end{aligned}$$

2. For all $x, x' \in L_-$ and for all $y \in L_+$ we have:

$$\{[x,x'],y\} = \{x, \psi_+(x')(y)\} - \{x', \psi_+(x)(y)\}$$

For all $x \in L_-$ and for all $y, y' \in L_+$ we have:

$$\{x,[y,y']\} = \{\psi_-(y')(x), y\} - \{\psi_-(y)(x), y'\}$$

3. For all $x \in L_-$, for all $y \in L_+$ and for all $h \in L_0$ we have:

$$\psi_-(\phi_+(h)(y))(x) = \phi_-(h)(\psi_-(y)(x)) - \psi_-(y)(\phi_-(h)(x))$$

$$\psi_+(\phi_-(h)(x))(y) = \phi_+(h)(\psi_+(x)(y)) - \psi_+(x)(\phi_+(h)(y))$$

4. For all $x \in L_-$, for all $y \in L_+$ and for all $h \in L_0$ we have:

$$[h,\{x,y\}] = \{\phi_-(h)(x), y\} + \{x, \phi_+(h)(y)\}$$

In this situation, we can carry out the following construction:

Theorem Let $L := L_- \oplus L_0 \oplus L_+$. Then L is a Lie algebra with the bracket:

$$\begin{aligned}&[(x,h,y),(x',h',y')] =\\ &([x,x'] + (\phi_-(h)(x') - \phi_-(h')(x)) + (\psi_-(y)(x') - \psi_-(y')(x)),\\ &[h,h'] + (\{x',y\} - \{x,y'\}),\\ &[y,y'] + (\phi_+(h)(y') - \phi_+(h')(y)) + (\psi_+(x)(y') - \psi_+(x')(y)))\end{aligned}$$

Proof. The bracket is obviously antisymmetric. We now verify the Jacobi identity. The expression $[[(x,h,y),(x',h',y')],(x'',h'',y'')]$ has three components. The L_--component is:

$$\begin{aligned}[[x,x'],x''] &+ [\phi_-(h)(x'),x''] - [\phi_-(h')(x),x'']\\ &+ [\psi_-(y)(x'),x''] - [\psi_-(y')(x),x'']\\ &+ \phi_-([h,h'])(x'') + \phi_-(\{x',y\})(x'') - \phi_-(\{x,y'\})(x'')\\ &- \phi_-(h'')([x,x']) - \phi_-(h'')(\phi_-(h)(x')) + \phi_-(h'')(\phi_-(h')(x))\\ &- \phi_-(h'')(\psi_-(y)(x')) + \phi_-(h'')(\psi_-(y')(x))\\ &+ \psi_-([y,y'])(x'') + \psi_-(\phi_+(h)(y'))(x'') - \psi_-(\phi_+(h')(y))(x'')\\ &+ \psi_-(\psi_+(x)(y'))(x'') - \psi_-(\psi_+(x')(y))(x'')\\ &- \psi_-(y'')([x,x']) - \psi_-(y'')(\phi_-(h)(x')) + \psi_-(y'')(\phi_-(h')(x))\\ &- \psi_-(y'')(\psi_-(y)(x')) + \psi_-(y'')(\psi_-(y')(x))\end{aligned}$$

If we now form the cyclic sum

$$[[(x,h,y),(x',h',y')],(x'',h'',y'')]+[[(x',h',y'),(x'',h'',y'')],(x,h,y)]$$
$$+[[(x'',h'',y''),(x,h,y)],(x',h',y')],$$

the first of the above terms vanishes by the Jacobi identity in L_-. Since the cyclic sum remains unchanged if we perform a cyclic permutation in some summand, the second, third and ninth term add up to zero, since ϕ_- maps into the derivations. Similarly, the fourth, fifth, seventh, eighth, seventeenth, eighteenth and nineteenth term vanish by the first compatibility assumption. By the same procedure, we can show that the cyclic sum of the sixth, tenth and eleventh term, the cyclic sum of the twelfth, fifteenth and twenty-first term, the cyclic sum of the thirteenth, sixteenth and twentieth term as well as the cyclic sum of the twenty-second and the twenty-third term vanish. We now turn to the L_0-component. It reads:

$$\begin{aligned}
&[[h,h'],h'']+[\{x',y\},h'']-[\{x,y'\},h'']+\\
&\{x'',[y,y']\}+\{x'',\phi_+(h)(y')\}-\{x'',\phi_+(h')(y)\}+\\
&\{x'',\psi_+(x)(y')\}-\{x'',\psi_+(x')(y)\}-\\
&\{[x,x'],y''\}-\{\phi_-(h)(x'),y''\}+\{\phi_-(h')(x),y''\}-\\
&\{\psi_-(y)(x'),y''\}+\{\psi_-(y')(x),y''\}
\end{aligned}$$

If we form again the cyclic sum, the first term vanishes by the Jacobi identity in L_0. The second, fifth and eleventh term vanish by the fourth compatibility assumption, as well as the third, sixth and tenth term. By the second compatibility assumption, the cyclic sum of the seventh, eighth and ninth term is zero, as well as the cyclic sum of the fourth, twelfth and thirteenth term. That the Jacobi identity also holds in the L_+-component is shown exactly as for the L_--component.$\square$.

3.3 We explain now which Lie algebras have the form of the above construction.

Theorem Suppose that L is a Lie algebra which is the direct sum of three Lie subalgebras L_-, L_0 and L_+:

$$L = L_- \oplus L_0 \oplus L_+$$

Suppose further that the vector spaces $L_- \oplus L_0$ and $L_0 \oplus L_+$ are Lie subalgebras which contain L_- resp. L_+ as ideals. Then there are mappings

$$\phi_+ : L_0 \to Der(L_+) \quad \phi_- : L_0 \to Der(L_-)$$

$$\psi_+ : L_- \to End(L_+) \quad \psi_- : L_+ \to End(L_-)$$

$$\{\cdot,\cdot\} : L_- \times L_+ \to L_0, (x,y) \mapsto \{x,y\}$$

that satisfy the conditions of Theorem 3.2 such that the Lie bracket can be written as in that Theorem.

Proof. Suppose that $p_- : L \to L_-, p_+ : L \to L_+$ and $p_0 : L \to L_0$ are the projections determined by the above decomposition. We define the mappings by the following equations:

$$\begin{aligned}
\phi_-(h)(x) &= p_-([(0,h,0),(x,0,0)]) \\
\phi_+(h)(y) &= p_+([(0,h,0),(0,0,y)]) \\
\psi_-(y)(x) &= p_-([(0,0,y),(x,0,0)]) \\
\psi_+(x)(y) &= p_+([(x,0,0),(0,0,y)]) \\
\{x,y\} &= p_0([(0,0,y),(x,0,0)])
\end{aligned}$$

We now prove that Lie bracket of L has the form given in Theorem 3.2. We begin with the L_--component:

$$\begin{aligned}
&p_-([(x,h,y),(x',h',y')]) = \\
&\quad p_-([(x,0,0),(x',0,0)]) + p_-([(x,0,0),(0,h',0)]) + p_-([(x,0,0),(0,0,y')]) + \\
&\quad p_-([(0,h,0),(x',0,0)]) + p_-([(0,h,0),(0,h',0)]) + p_-([(0,h,0),(0,0,y')]) + \\
&\quad p_-([(0,0,y),(x',0,0)]) + p_-([(0,0,y),(0,h',0)]) + p_-([(0,0,y),(0,0,y')])
\end{aligned}$$

In the last expression, the fifth, sixth, eighth and ninth term vanish by the assumption that $L_0 \oplus L_+$ is a Lie subalgebra, and we are left with:

$$\begin{aligned}
&p_-([(x,h,y),(x',h',y')]) = \\
&\qquad [x,x'] + \phi_-(h)(x') - \phi_-(h')(x) + \psi_-(y)(x') - \psi_-(y')(x)
\end{aligned}$$

By similar calculations one arrives at the form of the other two components. The Jacobi identity for L implies that the compatibility conditions in Theorem 3.2 hold. As an example, we prove the first part of the third condition:

$$\begin{aligned}
\{[x,x'],y\} &= p_0([(0,0,y),[(x,0,0),(x',0,0)]]) \\
&= p_0([[(0,0,y),(x,0,0)],(x',0,0)]) + p_0([(x,0,0),[(0,0,y),(x',0,0)]]) \\
&= p_0([(\psi_-(y)(x),\{x,y\},-\psi_+(x)(y)),(x',0,0)]) \\
&\qquad + p_0([(x,0,0),(\psi_-(y)(x'),\{x',y\},-\psi_+(x')(y))]) \\
&= -\{x',\psi_+(x)(y)\} + \{x,\psi_+(x')(y)\}
\end{aligned}$$

The other conditions can be verified similarly. $\Box$

3.4 Of course, there is an analogue of the first construction for Lie coalgebras. A very special case hereof will be sufficient for our later purposes. We shall call the Lie coalgebra structure that is obtained in the following Proposition the two-sided cosemidirect product of L_-, L_0 and L_+.

Proposition Suppose that L_-, L_+ and L_0 are Lie coalgebras. Suppose that L_+ is a positive L_0-Lie comodule via $\delta_+ : L_+ \to L_0 \otimes L_+$, and that L_- is a negative L_0-comodule via $\delta_- : L_- \to L_0 \otimes L_-$. Suppose in addition that δ_+ and δ_- are coderivations. Then $L_- \oplus L_0 \oplus L_+$ is a Lie coalgebra with the coproduct:

$$\begin{aligned}
&\Delta((x,h,y)) = \\
&\quad (x_1,0,0)\otimes(x_2,0,0) - (0,x^1,0)\otimes(x^2,0,0) + (x^2,0,0)\otimes(0,x^1,0) \\
&\qquad + (0,h_1,0)\otimes(0,h_2,0) \\
&\qquad - (0,0,y_1)\otimes(0,0,y_2) + (0,y^1,0)\otimes(0,0,y^2) - (0,0,y^2)\otimes(0,y^1,0)
\end{aligned}$$

Proof. By definition, we have to prove that Δ induces a Lie algebra structure on the dual vector space. $L_-^* \oplus L_0^* \oplus L_+^*$ can be identified with $(L_- \oplus L_0 \oplus L_+)^*$ via the isomorphism:

$$\begin{gathered}
L_-^* \oplus L_0^* \oplus L_+^* \to (L_- \oplus L_0 \oplus L_+)^* \\
(x^*,h^*,y^*) \mapsto ((x,h,y) \mapsto x^*(x) + h^*(h) + y^*(y))
\end{gathered}$$

Define

$$\phi_+ : L_0^* \to Der(L_+^*) \quad \phi_- : L_0^* \to Der(L_-^*)$$

by $(\phi_+(h^*)(y^*))(y) = h^*(y^1)y^*(y^2)$ and $(\phi_-(h^*)(x^*))(x) = -h^*(x^1)x^*(x^2)$. It follows from Proposition 2.1 that $\phi_+(h^*)$ and $\phi_-(h^*)$ in fact are derivations and that ϕ_+ and ϕ_- are Lie algebra homomorphisms. Using this notation, the induced Lie bracket on the dual vector space can be written in the following form:

$$\begin{aligned}
&[(x^*,h^*,y^*),(x'^*,h'^*,y'^*)](x,h,y) \\
&= ((x^*,h^*,y^*)\otimes(x'^*,h'^*,y'^*))((x_1,0,0)\otimes(x_2,0,0) \\
&\qquad - (0,x^1,0)\otimes(x^2,0,0) + (x^2,0,0)\otimes(0,x^1,0) + (0,h_1,0)\otimes(0,h_2,0) \\
&\qquad - (0,0,y_1)\otimes(0,0,y_2) + (0,y^1,0)\otimes(0,0,y^2) - (0,0,y^2)\otimes(0,y^1,0)) \\
&= x^*(x_1)x'^*(x_2) - h^*(x^1)x'^*(x^2) + x^*(x^2)h'^*(x^1) \\
&\qquad + h^*(h_1)h'^*(h_2) - y^*(y_1)y'^*(y_2) + h^*(y^1)y'^*(y^2) - y^*(y^2)h'^*(y^1) \\
&= ([x^*,x'^*] + \phi_-(h^*)(x'^*) - \phi_-(h'^*)(x^*), [h^*,h'^*], \\
&\qquad - [y^*,y'^*] + \phi_+(h^*)(y'^*) - \phi_+(h'^*)(y^*))(x,h,y)
\end{aligned}$$

We see that this bracket has the form of the first construction, where ψ_+, ψ_- and $\{\cdot,\cdot\}$ are equal to zero and the Lie bracket of L_+^* is replaced by its negative. It is easy to see that the compatibility conditions above are satisfied in this situation, therefore proving that we have defined a Lie algebra structure on the dual vector space.$\square$

3.5 The first construction contains as special case a construction that involves two Lie algebras which occur as summands of a larger one. This construction, which is attributed to M. Takeuchi in [2], is called the bismash product. Suppose that L_- and L_+ are two Lie algebras. Suppose further that there are two Lie algebra morphisms:

$$\psi_+ : L_- \to End(L_+) \quad \psi_- : L_+ \to End(L_-)$$

such that the following conditions are satisfied: For all $x, x' \in L_-$ and for all $y \in L_+$ we have:

$$\begin{aligned}\psi_-(y)([x,x']) =& [\psi_-(y)(x), x'] + [x, \psi_-(y)(x')] \\ &+ \psi_-(\psi_+(x')(y))(x) - \psi_-(\psi_+(x)(y))(x')\end{aligned}$$

For all $x \in L_-$ and for all $y, y' \in L_+$ we have:

$$\begin{aligned}\psi_+(x)([y,y']) =& [\psi_+(x)(y), y'] + [y, \psi_+(x)(y')] \\ &+ \psi_+(\psi_-(y')(x))(y) - \psi_+(\psi_-(y)(x))(y')\end{aligned}$$

In this situation, we can carry out the following construction:

Theorem Let $L := L_- \oplus L_+$. Then L is a Lie algebra with the bracket:

$$\begin{aligned}&[(x,y),(x',y')] = \\ &\quad ([x,x'] + (\psi_-(y)(x') - \psi_-(y')(x)), [y,y'] + (\psi_+(x)(y') - \psi_+(x')(y)))\end{aligned}$$

Proof. This is simply the case of the first construction where $L_0 = \{0\}$. Therefore, ϕ_+, ϕ_- and $\{\cdot, \cdot\}$ are also equal to zero. □

3.6 The construction above is closely interrelated with the first construction in [15], [16]. Namely, the enveloping algebra of L can be constructed from the enveloping algebras of L_-, L_0 and L_+ by the first construction in [15]. This will not be needed in the sequel, and we shall only outline the construction. Denote the enveloping algebras by $A = \mathcal{U}(L_-), H = \mathcal{U}(L_0)$ and $B = \mathcal{U}(L_+)$. We turn A and B into Yetter-Drinfel'd modules in the following way: The derivation $\phi_-(h)$ for $h \in L_0$ extends uniquely to a derivation of the enveloping algebra $A = \mathcal{U}(L_-)$. In this way, we get a Lie algebra homomorphism from L_0 to $Der(A)$ which by the universal property of enveloping algebras yields an algebra map from $H = \mathcal{U}(L_0)$ to $End(A)$ that gives rise to a left H-module structure on A. Choosing the trivial left H-comodule structure one can check that we have constructed a Yetter-Drinfel'd module structure on A which obviously satisfies

$$h \rightharpoonup x = \phi_-(h)(x)$$

whenever it happens that $h \in L_0$ and $x \in L_-$, where the arrow is used to indicate the action of an element $h \in H$ on an element $x \in A$. By a similar reasoning we can endow B with the structure of a right Yetter-Drinfel'd module that satisfies

$$y \leftharpoonup h = -\phi_+(h)(y)$$

for $h \in L_0$ and $y \in L_+$.

It is slightly more difficult to prove that there are unique mappings

$$\rightharpoonup: B \otimes A \to A \qquad \leftharpoonup: B \otimes A \to B \qquad \sharp : B \otimes A \to H$$

that satisfy the compatibility conditions listed in [15] and are determined on generators by the equations:

$$y \rightharpoonup x = \psi_-(y)(x) \qquad y \leftharpoonup x = -\psi_+(y)(x) \qquad y\sharp x = \{x, y\}$$

for $x \in L_-$ and $y \in L_+$. After this has been established, we can introduce a coproduct and a product on $A \otimes H \otimes B$ by the equations

$$\Delta(a \otimes h \otimes b) = (a_1 \otimes {a_2}^1 h_1 \otimes {b_1}^1) \otimes ({a_2}^2 \otimes h_2 {b_1}^2 \otimes b_2)$$

$$(a \otimes h \otimes b)(a' \otimes h' \otimes b') =$$
$$a(h_1 \rightarrow ({b_1}^1 \rightharpoonup a'_1)) \otimes h_2 {b_1}^2 (b_2 \sharp a'_2) {a'_3}^1 h'_1 \otimes ((b_3 \leftharpoonup {a'_3}^2) \leftarrow h'_2) b'$$

This turns $A \otimes H \otimes B$ into a Hopf algebra (cf. [15]). The canonical mapping

$$L_- \oplus L_0 \oplus L_+ \to A \otimes H \otimes B$$

is a Lie algebra homomorphism and therefore extends to an algebra homomorphism

$$\mathcal{U}(L_- \oplus L_0 \oplus L_+) \to A \otimes H \otimes B$$

that turns out to be a Hopf algebra isomorphism.

4 The second construction

4.1 In this Section, we attach under some technical assumptions to a pair of Yetter-Drinfel'd Lie algebras which are paired via a Lie algebra biform a Lie algebra of the form of the first construction. We shall describe in the next Section a cohomological procedure to construct such pairs of Yetter-Drinfel'd Lie algebras, and then see that symmetrizable Kac-Moody algebras are of this form.

4.2 The following second construction is fundamental. We shall carry it out in somewhat greater generality than necessary.

Theorem Suppose that L_0 is a Lie algebra. Suppose that L_+ and L_- are Yetter-Drinfel'd Lie algebras over L_0 and that

$$\langle \cdot, \cdot \rangle : L_- \times L_+ \to K$$

is a Lie algebra biform. We suppose further that we are given a linear map

$$\Delta_0 : L_0 \to L_0 \otimes L_0$$

which is compatible with the maps $\delta_+ : L_+ \to L_0 \otimes L_+$ and $\delta_- : L_- \to L_0 \otimes L_-$ in the following way:

$$\delta_+(\phi_+(h)(y)) = [h, y^1] \otimes y^2 + y^1 \otimes \phi_+(h)(y^2) + h_1 \otimes \phi_+(h_2)(y)$$
$$\delta_-(\phi_-(h)(x)) = [h, x^1] \otimes x^2 + x^1 \otimes \phi_-(h)(x^2) - h_1 \otimes \phi_-(h_2)(x)$$

for all $h \in L_0, x \in L_-$ and $y \in L_+$. Furthermore, we assume that we have for all $x \in L_-$ and $y \in L_+$:

$$x^2 \otimes \phi_+(x^1)(y) = \phi_-(y^1)(x) \otimes y^2$$

Define:

$$\{\cdot,\cdot\} : L_- \times L_+ \to L_0, (x,y) \mapsto \{x,y\} := y^1\langle x, y^2\rangle - x^1\langle x^2, y\rangle$$

Denote the coadjoint action corresponding to the Lie algebra biform by

$$\psi_+ : L_- \to End(L_+) \quad \text{resp.} \quad \psi_- : L_+ \to End(L_-)$$

Then $L_- \oplus L_0 \oplus L_+$ becomes a Lie algebra with the Lie bracket:

$$\begin{aligned}
&[(x,h,y),(x',h',y')] = \\
&([x,x'] + (\phi_-(h)(x') - \phi_-(h')(x)) + (\psi_-(y)(x') - \psi_-(y')(x)), \\
&[h,h'] + (\{x',y\} - \{x,y'\}), \\
&[y,y'] + (\phi_+(h)(y') - \phi_+(h')(y)) + (\psi_+(x)(y') - \psi_+(x')(y)))
\end{aligned}$$

Proof. Since this Theorem is important, we shall give some details of the proof. We proceed in steps.

1. We first need an auxiliary calculation:

$$\begin{aligned}
\psi_-(\psi_+(x)(y))(x') &= \langle x, y_2\rangle \psi_-(y_1)(x') \\
&= \langle x, y_2\rangle\langle x'_2, y_1\rangle x'_1 = \langle [x'_2, x], y\rangle x'_1
\end{aligned}$$

   The equation $\psi_+(\psi_-(y)(x))(y') = \langle x, [y'_2, y]\rangle y'_1$ can be proved by a similar reasoning.

2. We now verify the first part of condition 3.2.1:

$$\begin{aligned}
&\psi_-(y)([x,x']) - [\psi_-(y)(x), x'] - [x, \psi_-(y)(x')] \\
&\qquad - \psi_-(\psi_+(x')(y))(x) + \psi_-(\psi_+(x)(y))(x') \\
&= [x,x']_1\langle [x,x']_2, y\rangle - \langle x_2, y\rangle[x_1, x'] - \langle x'_2, y\rangle[x, x'_1] \\
&\qquad - \langle [x_2, x'], y\rangle x_1 + \langle [x'_2, x], y\rangle x'_1 \\
&= x^2\langle \phi_-(x^1)(x'), y\rangle - \phi_-(x^1)(x')\langle x^2, y\rangle \\
&\qquad + \phi_-(x'^1)(x)\langle x'^2, y\rangle - x'^2\langle \phi_-(x'^1)(x), y\rangle \\
&= -x^2\langle x', \phi_+(x^1)(y)\rangle - \phi_-(x^1)(x')\langle x^2, y\rangle \\
&\qquad + \phi_-(x'^1)(x)\langle x'^2, y\rangle + x'^2\langle x, \phi_+(x'^1)(y)\rangle \\
&= -\phi_-(y^1)(x)\langle x', y^2\rangle - \phi_-(x^1)(x')\langle x^2, y\rangle \\
&\qquad + \phi_-(x'^1)(x)\langle x'^2, y\rangle + \phi_-(y^1)(x')\langle x, y^2\rangle \\
&= \phi_-(\{x,y\})(x') - \phi_-(\{x',y\})(x)
\end{aligned}$$

   The second part of condition 3.2.1 can be shown similarly.

3. We now proceed to verify the second part of condition 3.2.2, leaving the first part to the reader. We first calculate the left hand side:

$$
\begin{aligned}
\{x,[y,y']\} &= [y,y']^1\langle x,[y,y']^2\rangle - x^1\langle x^2,[y,y']\rangle \\
&= y^1\langle x,[y^2,y']\rangle + y'^1\langle x,[y,y'^2]\rangle - x^1\langle x^2{}_1,y\rangle\langle x^2{}_2,y'\rangle \\
&= y^1\langle x_1,y^2\rangle\langle x_2,y'\rangle + y'^1\langle x_1,y\rangle\langle x_2,y'^2\rangle \\
&\quad - x_1{}^1\langle x_1{}^2,y\rangle\langle x_2,y'\rangle - x_2{}^1\langle x_1,y\rangle\langle x_2{}^2,y'\rangle
\end{aligned}
$$

On the other hand, we have:

$$
\begin{aligned}
\{\psi_-(y')(x),y\} - \{\psi_-(y)(x),y'\} =\ & \langle x_2,y'\rangle\{x_1,y\} - \langle x_2,y\rangle\{x_1,y'\} \\
=\ & \langle x_2,y'\rangle\langle x_1,y^2\rangle y^1 - \langle x_2,y'\rangle\langle x_1{}^2,y\rangle x_1{}^1 \\
& - \langle x_2,y\rangle\langle x_1,y'^2\rangle y'^1 + \langle x_2,y\rangle\langle x_1{}^2,y'\rangle x_1{}^1
\end{aligned}
$$

Both expressions are equal due to the skew-symmetry of the cobracket.

4. We now verify condition 3.2.3:

$$
\begin{aligned}
&\psi_-(\phi_+(h)(y))(x) \\
&= x_1\langle x_2,\phi_+(h)(y)\rangle = -x_1\langle\phi_-(h)(x_2),y\rangle \\
&= \phi_-(h)(x_1)\langle x_2,y\rangle - (\phi_-(h)(x))_1\langle(\phi_-(h)(x))_2,y\rangle \\
&= \phi_-(h)(\psi_-(y)(x)) - \psi_-(y)(\phi_-(h)(x))
\end{aligned}
$$

The second part can be proved similarly.

5. Finally, we prove condition 3.2.4:

$$
\begin{aligned}
&\{x,\phi_+(h)(y)\} + \{\phi_-(h)(x),y\} \\
&\quad= (\phi_+(h)(y))^1\langle x,(\phi_+(h)(y))^2\rangle + y^1\langle\phi_-(h)(x),y^2\rangle \\
&\qquad - (\phi_-(h)(x))^1\langle(\phi_-(h)(x))^2,y\rangle - x^1\langle x^2,\phi_+(h)(y)\rangle \\
&\quad= (\phi_+(h)(y))^1\langle x,(\phi_+(h)(y))^2\rangle - y^1\langle x,\phi_+(h)(y^2)\rangle \\
&\qquad - (\phi_-(h)(x))^1\langle(\phi_-(h)(x))^2,y\rangle + x^1\langle\phi_-(h)(x^2),y\rangle \\
&\quad= [h,y^1]\langle x,y^2\rangle + y^1\langle x,\phi_+(h)(y^2)\rangle + h_1\langle x,\phi_+(h_2)(y)\rangle \\
&\qquad - y^1\langle x,\phi_+(h)(y^2)\rangle - [h,x^1]\langle x^2,y\rangle \\
&\qquad - x^1\langle\phi_-(h)(x^2),y\rangle + h_1\langle\phi_-(h_2)(x),y\rangle + x^1\langle\phi_-(h)(x^2),y\rangle \\
&\quad= [h,y^1]\langle x,y^2\rangle - [h,x^1]\langle x^2,y\rangle = [h,\{x,y\}]
\end{aligned}
$$

This completes the proof. □

4.3 So far, we have not used any compatibility conditions between the coproduct Δ_0 of L_0 and the mappings δ_+ and δ_-. However, if we impose such assumptions, something more can be said: $L_- \oplus L_0 \oplus L_+$ becomes a Lie bialgebra.

Theorem Suppose that L_0 is a Lie bialgebra. Suppose that L_+ is a positive Yetter-Drinfel'd Lie algebra and that L_- is a negative Yetter-Drinfel'd Lie algebra over L_0. Suppose that

$$\langle \cdot, \cdot \rangle : L_- \times L_+ \to K$$

is a Lie algebra biform. Furthermore, we assume that we have for all $x \in L_-$ and $y \in L_+$:

$$x^2 \otimes \phi_+(x^1)(y) = \phi_-(y^1)(x) \otimes y^2$$

Define:

$$\{\cdot, \cdot\} : L_- \times L_+ \to L_0, (x, y) \mapsto \{x, y\} := y^1 \langle x, y^2 \rangle - x^1 \langle x^2, y \rangle$$

Denote the coadjoint action corresponding to the Lie algebra biform by

$$\psi_+ : L_- \to End(L_+) \quad \text{resp.} \quad \psi_- : L_+ \to End(L_-)$$

Then $L_- \oplus L_0 \oplus L_+$ becomes a Lie bialgebra with the Lie bracket:

$$\begin{aligned}
&[(x, h, y), (x', h', y')] = \\
&([x, x'] + (\phi_-(h)(x') - \phi_-(h')(x)) + (\psi_-(y)(x') - \psi_-(y')(x)), \\
&[h, h'] + (\{x', y\} - \{x, y'\}), \\
&[y, y'] + (\phi_+(h)(y') - \phi_+(h')(y)) + (\psi_+(x)(y') - \psi_+(x')(y)))
\end{aligned}$$

and the two-sided cosemidirect cobracket:

$$\begin{aligned}
\Delta((x, h, y)) = &\\
(x_1, 0, 0) \otimes (x_2, 0, 0) &- (0, x^1, 0) \otimes (x^2, 0, 0) + (x^2, 0, 0) \otimes (0, x^1, 0) \\
&+ (0, h_1, 0) \otimes (0, h_2, 0) \\
&- (0, 0, y_1) \otimes (0, 0, y_2) + (0, y^1, 0) \otimes (0, 0, y^2) - (0, 0, y^2) \otimes (0, y^1, 0)
\end{aligned}$$

Proof. For the purposes of this proof, we shall adopt a shorthand notation. We abbreviate: $x = (x, 0, 0), h = (0, h, 0), y = (0, 0, y)$ for $x \in L_-, h \in L_0$ and $y \in L_+$. Note that this identification is inaccurate since, for example, the canonical mapping $L_- \to L_- \oplus L_0 \oplus L_+$ is not a Lie coalgebra homomorphism. Therefore, the term x_1 could mean $(x_1, 0, 0)$ or $(x, 0, 0)_1$. We now overcome this ambiguity for purposes of this proof by agreeing on the first possibility.

1. We have for $x, x' \in L_-$:

$$\begin{aligned}
\Delta([x, x']) &= [x, x']_1 \otimes [x, x']_2 - [x, x']^1 \otimes [x, x']^2 + [x, x']^2 \otimes [x, x']^1 \\
&= [x, x'_1] \otimes x'_2 + x'_1 \otimes [x, x'_2] - [x', x_1] \otimes x_2 - x_1 \otimes [x', x_2] \\
&\quad + x^2 \otimes \phi_-(x^1)(x') - \phi_-(x^1)(x') \otimes x^2 \\
&\quad + \phi_-(x'^1)(x) \otimes x'^2 - x'^2 \otimes \phi_-(x'^1)(x) \\
&\quad - x^1 \otimes [x^2, x'] - x'^1 \otimes [x, x'^2] \\
&\quad + [x^2, x'] \otimes x^1 + [x, x'^2] \otimes x'^1
\end{aligned}$$

On the other hand, we have:

$$\begin{aligned} x.\Delta(x') &= [x, x'_1] \otimes x'_2 - [x, x'^1] \otimes x'^2 + [x, x'^2] \otimes x'^1 \\ &\quad + x'_1 \otimes [x, x'_2] - x'^1 \otimes [x, x'^2] + x'^2 \otimes [x, x'^1] \\ &= [x, x'_1] \otimes x'_2 + \phi_-(x'^1)(x) \otimes x'^2 + [x, x'^2] \otimes x'^1 \\ &\quad + x'_1 \otimes [x, x'_2] - x'^1 \otimes [x, x'^2] - x'^2 \otimes \phi_-(x'^1)(x) \end{aligned}$$

From this, we conclude that $\Delta([x, x']) = x.\Delta(x') - x'.\Delta(x)$, which according to our shorthand notation means that $\Delta([(x, 0, 0), (x', 0, 0)]) = (x, 0, 0).\Delta((x', 0, 0)) - (x', 0, 0).\Delta((x, 0, 0))$. One can show similarly that $\Delta([y, y']) = y.\Delta(y') - y'.\Delta(y)$. It is clear that we have $\Delta([h, h']) = h.\Delta(h') - h'.\Delta(h)$, because the canonical injection $L_0 \to L_- \otimes L_0 \otimes L_+$ is a Lie algebra and a Lie coalgebra homomorphism.

2. We have for $y \in L_+$ and $h \in L_0$:

$$\begin{aligned} \Delta([h, y]) =& \Delta(\phi_+(h)(y)) \\ =& - \phi_+(h)(y_1) \otimes y_2 - y_1 \otimes \phi_+(h)(y_2) \\ &+ [h, y^1] \otimes y^2 + y^1 \otimes \phi_+(h)(y^2) + h_1 \otimes \phi_+(h_2)(y) \\ &- y^2 \otimes [h, y^1] - \phi_+(h)(y^2) \otimes y^1 - \phi_+(h_2)(y) \otimes h_1 \end{aligned}$$

On the other hand, we have:

$$\begin{aligned} h.\Delta(y) - y.\Delta(h) =& - [h, y_1] \otimes y_2 - y_1 \otimes [h, y_2] \\ &+ [h, y^1] \otimes y^2 + y^1 \otimes [h, y^2] - [h, y^2] \otimes y^1 \\ &- y^2 \otimes [h, y^1] - [y, h_1] \otimes h_2 - h_1 \otimes [y, h_2] \\ =& - \phi_+(h)(y_1) \otimes y_2 - y_1 \otimes \phi_+(h)(y_2) \\ &+ [h, y^1] \otimes y^2 + y^1 \otimes \phi_+(h)(y^2) - \phi_+(h)(y^2) \otimes y^1 \\ &- y^2 \otimes [h, y^1] + \phi_+(h_1)(y) \otimes h_2 + h_1 \otimes \phi_+(h_2)(y) \end{aligned}$$

The comparison of both expressions shows that they are equal. By a similar calculation, it follows that $\Delta([x, h]) = x.\Delta(h) - h.\Delta(x)$.

3. Finally, we verify the most difficult of these equations, which is $\Delta([x, y]) = x.\Delta(y) - y.\Delta(x)$ for $x \in L_-, y \in L_+$. We have:

$$\begin{aligned} &x.\Delta(y) - y.\Delta(x) = \\ &\quad - [x, y_1] \otimes y_2 - y_1 \otimes [x, y_2] + [x, y^1] \otimes y^2 + y^1 \otimes [x, y^2] \\ &\quad - [x, y^2] \otimes y^1 - y^2 \otimes [x, y^1] - [y, x_1] \otimes x_2 - x_1 \otimes [y, x_2] \\ &\quad + [y, x^1] \otimes x^2 + x^1 \otimes [y, x^2] - [y, x^2] \otimes x^1 - x^2 \otimes [y, x^1] \end{aligned}$$

Here, the third and last term cancel, as well as the sixth and the ninth. Expanding the remaining eight terms yields:

$$
\begin{aligned}
&\psi_-(y_1)(x) \otimes y_2 + \{x, y_1\} \otimes y_2 - \psi_+(x)(y_1) \otimes y_2 \\
&+ y_1 \otimes \psi_-(y_2)(x) + y_1 \otimes \{x, y_2\} - y_1 \otimes \psi_+(x)(y_2) \\
&- y^1 \otimes \psi_-(y^2)(x) - y^1 \otimes \{x, y^2\} + y^1 \otimes \psi_+(x)(y^2) \\
&+ \psi_-(y^2)(x) \otimes y^1 + \{x, y^2\} \otimes y^1 - \psi_+(x)(y^2) \otimes y^1 \\
&- \psi_-(y)(x_1) \otimes \; x_2 - \{x_1, y\} \otimes x_2 + \psi_+(x_1)(y) \otimes x_2 \\
&- x_1 \otimes \psi_-(y)(x_2) - x_1 \otimes \{x_2, y\} + x_1 \otimes \psi_+(x_2)(y) \\
&+ x^1 \otimes \psi_-(y)(x^2) + x^1 \otimes \{x^2, y\} - x^1 \otimes \psi_+(x^2)(y) \\
&- \psi_-(y)(x^2) \otimes x^1 - \{x^2, y\} \otimes x^1 + \psi_+(x^2)(y) \otimes x^1
\end{aligned}
$$

In this expression, the first and the eighteenth as well as the fourth and the fifteenth term cancel. Expanding further, we obtain

$$
\begin{aligned}
&\langle x, {y_1}^2\rangle {y_1}^1 \otimes y_2 - \langle x^2, y_1\rangle x^1 \otimes y_2 - \langle x, y_{12}\rangle y_{11} \otimes y_2 + \langle x, {y_2}^2\rangle y_1 \otimes {y_2}^1 \\
&- \langle x^2, y_2\rangle y_1 \otimes x^1 - \langle x, y_{22}\rangle y_1 \otimes y_{21} - \langle x_2, y^2\rangle y^1 \otimes x_1 - \langle x, y^{22}\rangle y^1 \otimes y^{21} \\
&+ \langle x^2, y^2\rangle y^1 \otimes x^1 + \langle x, {y^2}_2\rangle y^1 \otimes {y^2}_1 + \langle x_2, y^2\rangle x_1 \otimes y^1 + \langle x, y^{22}\rangle y^{21} \otimes y^1 \\
&- \langle x^2, y^2\rangle x^1 \otimes y^1 - \langle x, {y^2}_2\rangle {y^2}_1 \otimes y^1 - \langle x_{12}, y\rangle x_{11} \otimes x_2 - \langle x_1, y^2\rangle y^1 \otimes x_2 \\
&+ \langle {x_1}^2, y\rangle {x_1}^1 \otimes x_2 - \langle x_{22}, y\rangle x_1 \otimes x_{21} - \langle x_2, y^2\rangle x_1 \otimes y^1 + \langle {x_2}^2, y\rangle x_1 \otimes {x_2}^1 \\
&+ \langle {x^2}_2, y\rangle x^1 \otimes {x^2}_1 + \langle x^2, y^2\rangle x^1 \otimes y^1 - \langle x^{22}, y\rangle x^1 \otimes x^{21} - \langle x^2, y_2\rangle x^1 \otimes y_1 \\
&- \langle {x^2}_2, y\rangle {x^2}_1 \otimes x^1 - \langle x^2, y^2\rangle y^1 \otimes x^1 + \langle x^{22}, y\rangle x^{21} \otimes x^1 + \langle x^2, y_2\rangle y_1 \otimes x^1
\end{aligned}
$$

Here, the second and the twenty-fourth, the fifth and the last, the seventh and the sixteenth, the ninth and the twenty-sixth, the eleventh and the nineteenth, and also the thirteenth and the twenty-second cancel. The remaining terms are equal to:

$$
\begin{aligned}
&\langle x_1, y\rangle x_{21} \otimes x_{22} - \langle x_1, y\rangle {x_2}^1 \otimes {x_2}^2 + \langle x_1, y\rangle {x_2}^2 \otimes {x_2}^1 + \langle x^2, y\rangle {x^1}_1 \otimes {x^1}_2 \\
&- \langle x, y^2\rangle {y^1}_1 \otimes {y^1}_2 - \langle x, y_2\rangle y_{11} \otimes y_{12} + \langle x, y_2\rangle {y_1}^1 \otimes {y_1}^2 - \langle x, y_2\rangle {y_1}^2 \otimes {y_1}^1
\end{aligned}
$$

But this is $\Delta(\langle x_1, y\rangle x_2 + x^1\langle x^2, y\rangle - y^1\langle x, y^2\rangle + y_1\langle x, y_2\rangle)$, which is just $\Delta([x, y])$. $\square$

4.4 For the rest of this Section, we shall discuss various special cases of the second construction. The simplest special case is the one where L_+ is zero:

Corollary Suppose that L_0 is a Lie bialgebra. Suppose that L_- is a negative Yetter-Drinfel'd Lie algebra over L_0. Then $L_- \oplus L_0$ becomes a Lie bialgebra with the Lie bracket of a semidirect product:

$$[(x, h), (x', h')] = ([x, x'] + \phi_-(h)(x') - \phi_-(h')(x), [h, h'])$$

and the cosemidirect cobracket:

$$\Delta((x,h)) =$$
$$(x_1,0)\otimes(x_2,0)-(0,x^1)\otimes(x^2,0)+(x^2,0)\otimes(0,x^1)+(0,h_1)\otimes(0,h_2)$$

4.5 The second construction contains as a special case the construction of the classical double due to V. G. Drinfel'd (cf. [5], p. 816, cf. also [4] and [2]).

Corollary Suppose that L_+ and L_- are Lie bialgebras. Suppose that

$$\langle\cdot,\cdot\rangle : L_- \times L_+ \to K$$

is a bilinear form with the following properties:

1. For all $x \in L_-$ and $y, y' \in L_+$ we have: $\langle x_1, y\rangle\langle x_2, y'\rangle = \langle x, [y, y']\rangle$
2. For all $x, x' \in L_-$ and $y \in L_+$ we have: $\langle x, y_1\rangle\langle x', y_2\rangle = \langle [x, x'], y\rangle$

Denote the coadjoint action corresponding to the Lie algebra biform by

$$\psi_+ : L_- \to End(L_+) \quad \text{resp.} \quad \psi_- : L_+ \to End(L_-)$$

Then $L_- \oplus L_+$ becomes a Lie bialgebra with the Lie bracket:

$$[(x,y),(x',y')] =$$
$$([x,x']+(\psi_-(y)(x')-\psi_-(y')(x)),[y,y']+(\psi_+(x)(y')-\psi_+(x')(y)))$$

and the cobracket:

$$\Delta((x,y)) = (x_1,0)\otimes(x_2,0)-(0,y_1)\otimes(0,y_2)$$

Proof. This is just the case of the second construction where $L_0 = \{0\}$. □

Note that in the case where L_- is finite dimensional, L_+ can be chosen to be the dual vector space and the bilinear form can be chosen to be the evaluation map. In this case, the Yetter-Drinfel'd Lie modules are, as in the deformed case, precisely the modules over the Drinfel'd double:

Proposition Suppose that L_+ is a finite dimensional Lie bialgebra with dual vector space $L_- := L_+^*$, and that V is a negative Yetter-Drinfel'd Lie module over L_+. Then the map

$$\phi : L_- \oplus L_+ \to End(V), (x,y) \mapsto (v \mapsto x(v^1)v^2 + \phi_+(y)(v))$$

is a Lie algebra homomorphism. Conversely, if $\phi : L_- \oplus L_+ \to End(V)$ is a Lie algebra homomorphism and $y_1,\dots,y_n$ is a basis of L_+ with dual basis $y_1^*,\dots,y_n^*$ in L_-, then V becomes a negative Yetter-Drinfel'd Lie module over L_+ by defining:

$$\phi_+ : L_+ \to End(V), y \mapsto (v \mapsto \phi(0,y)(v))$$

$$\delta_+ : V \to L_+ \otimes V, v \mapsto \sum_{i=1}^{n} y_i \otimes \phi(y_i^*,0)(v)$$

The proof is based on direct verification and is omitted.

4.6 It should be emphasized that the conditions on the interrelation of L_+ and L_- imposed in Theorem 4.3 are not those which interrelate canonically a vector space and its dual. This is because we have not assumed that the Lie algebra biform be strict, and because we have assumed that L_- be negative instead of positive. As in the deformed case, it remains an open problem to attach to a (finite dimensional) Yetter-Drinfel'd Lie algebra L_+ a canonical other Yetter-Drinfel'd Lie algebra in such a way that the conditions of the above Theorem are satisfied. However, we can do this in a special case where the Yetter-Drinfel'd Lie algebras are symmetric:

Corollary Suppose that L_0 is a coabelian Lie bialgebra. Suppose that L_+ and L_- are positive symmetric Yetter-Drinfel'd Lie algebras over L_0. Suppose that

$$\langle\cdot,\cdot\rangle : L_- \times L_+ \to K$$

is a nondegenerate strict Lie algebra biform. Define:

$$\{\cdot,\cdot\} : L_- \times L_+ \to L_0,\ (x,y) \mapsto \{x,y\} := y^1\langle x, y^2\rangle - x^1\langle x^2, y\rangle = 2\, y^1\langle x, y^2\rangle$$

Denote the coadjoint action corresponding to the Lie algebra biform by

$$\psi_+ : L_- \to End(L_+) \quad \text{resp.} \quad \psi_- : L_+ \to End(L_-)$$

Then $L_- \oplus L_0 \oplus L_+$ becomes a Lie bialgebra with the Lie bracket:

$$\begin{aligned}
&[(x,h,y),(x',h',y')] = \\
&([x,x'] + (\phi_-(h)(x') - \phi_-(h')(x)) + (\psi_-(y)(x') - \psi_-(y')(x)), \\
&[h,h'] + (\{x',y\} - \{x,y'\}), \\
&[y,y'] + (\phi_+(h)(y') - \phi_+(h')(y)) + (\psi_+(x)(y') - \psi_+(x')(y)))
\end{aligned}$$

and the two-sided cosemidirect cobracket:

$$\begin{aligned}
&\Delta((x,h,y)) = \\
&\quad (x_1,0,0)\otimes(x_2,0,0) - (0,x^1,0)\otimes(x^2,0,0) + (x^2,0,0)\otimes(0,x^1,0) \\
&\qquad - (0,0,y_1)\otimes(0,0,y_2) + (0,y^1,0)\otimes(0,0,y^2) - (0,0,y^2)\otimes(0,y^1,0)
\end{aligned}$$

Proof. Observe that positive and negative coactions coincide since we have assumed that L_0 is coabelian. Our assertions follow directly from Theorem 4.3 if we can prove that we have

$$x^2 \otimes \phi_+(x^1)(y) = \phi_-(y^1)(x) \otimes y^2$$

for $x \in L_-$ and $y \in L_+$. But this follows from our symmetry assumption:

$$\begin{aligned}
\langle x^2, y'\rangle\langle x', \phi_+(x^1)(y)\rangle &= -\langle x, y'^2\rangle\langle x', \phi_+(y'^1)(y)\rangle \\
&= -\langle x, \phi_+(y^1)(y')\rangle\langle x', y^2\rangle \\
&= \langle \phi_-(y^1)(x), y'\rangle\langle x', y^2\rangle
\end{aligned}$$

□

4.7 After discussing the symmetric case, we now turn to the skew-symmetric case. Here, the second construction can be carried out in two steps: We can first carry out a variant of the Drinfel'd double construction as discussed in Subsection 4.5 to obtain a negative Yetter-Drinfel'd Lie algebra structure on $L_- \oplus L_+$ and afterwards apply the construction discussed in Subsection 4.4. The following Proposition describes the first step:

Proposition 1 Suppose that L_0 is a Lie bialgebra. Suppose that L_+ is a positive Yetter-Drinfel'd Lie algebra and that L_- is a negative Yetter-Drinfel'd Lie algebra over L_0. Suppose that

$$\langle\cdot,\cdot\rangle : L_- \times L_+ \to K$$

is a Lie algebra biform. We assume that we have for all $x \in L_-$ and $y \in L_+$:

1. $y^1\langle x, y^2\rangle = x^1\langle x^2, y\rangle$
2. $x^2 \otimes \phi_+(x^1)(y) = \phi_-(y^1)(x) \otimes y^2$

Denote the coadjoint action corresponding to the Lie algebra biform by

$$\psi_+ : L_- \to End(L_+) \quad \text{resp.} \quad \psi_- : L_+ \to End(L_-)$$

Then $L := L_- \oplus L_+$ becomes a negative Yetter-Drinfel'd Lie algebra with the Lie bracket:

$$\begin{aligned}&[(x,y),(x',y')] = \\ &\quad ([x,x'] + \psi_-(y)(x') - \psi_-(y')(x), [y,y'] + \psi_+(x)(y') - \psi_+(x')(y))\end{aligned}$$

the cobracket:

$$\Delta((x,y)) = (x_1, 0) \otimes (x_2, 0) - (0, 0, y_1) \otimes (0, 0, y_2),$$

the action:

$$\begin{aligned}\phi : L_0 &\to Der(L_- \oplus L_+)\\ h &\mapsto ((x,y) \mapsto (\phi_-(h)(x), \phi_+(h)(y)))\end{aligned}$$

and the coaction:

$$\begin{aligned}\delta : L &\to L_0 \otimes L\\ (x,y) &\mapsto x^1 \otimes (x^2, 0) - y^1 \otimes (0, y^2)\end{aligned}$$

Proof. We first prove that L is a Lie algebra. Observe that our assumptions suffice to carry out the second construction in the form of Theorem 4.3 to obtain a Lie algebra $L_- \oplus L_0 \oplus L_+$. Our first assumption says precisely that we have $\{x, y\} = 0$ for all $x \in L_-$, $y \in L_+$, and therefore the map

$$L_- \oplus L_+ \to L_- \oplus L_0 \oplus L_+,\ (x,y) \mapsto (x, 0, y)$$

is a Lie algebra homomorphism. Therefore, L is isomorphic to a subalgebra of a Lie algebra and is therefore itself a Lie algebra.

Of the remaining conditions, there are three that are not very easy. We have to prove that $\phi(h)$ is a derivation:

$$\phi(h)([z,z']) = [\phi(h)(z), z'] + [x, \phi(h)(z')],$$

that δ is a 1-cocycle:

$$\delta([z,z']) = z^1 \otimes [z^2, z'] + z'^1 \otimes [z, z'^2]$$

and that the coboundary of the coproduct is determined the action and the coaction:

$$\begin{aligned}&\Delta([z,z']) - z.\Delta(z') + z'.\Delta(z) = \\ &\qquad z^2 \otimes \phi(z^1)(z') - \phi(z^1)(z') \otimes z^2 + \phi(z'^1)(z) \otimes z'^2 - z'^2 \otimes \phi(z'^1)(z)\end{aligned}$$

for all $z, z' \in L$. In all three cases we can assume that $z = (x, 0),\ z' = (0, y)$. We have:

$$\begin{aligned}\phi(h)([z,z']) &= (-\phi_-(h)(\psi_-(y)(x)), \phi_+(h)(\psi_+(x)(y))) \\ [\phi(h)(z), z'] &= (-\psi_-(y)(\phi_-(h)(x)), \psi_+(\phi_-(h)(x))(y)) \\ [z, \phi(h)(z')] &= (-\psi_-(\phi_+(h)(y))(x), \psi_+(x)(\phi_+(h)(y)))\end{aligned}$$

The assertion now follows from condition 3.2.3 which we have already verified in the proof of Theorem 4.3. We now prove that δ is a 1-cocycle. We have:

$$\begin{aligned}\delta([z,z']) &= \delta((-\psi_-(y)(x), \psi_+(x)(y))) \\ &= -{x_1}^1 \otimes ({x_1}^2, 0)\langle x_2, y\rangle - {y_1}^1 \otimes (0, {y_1}^2)\langle x, y_2\rangle\end{aligned}$$

On the other hand, we have by our first assumption:

$$\begin{aligned}&z.\delta(z') - z'.\delta(z) = \\ &-y^1 \otimes (-\psi_-(y^2)(x), \psi_+(x)(y^2)) - x^1 \otimes (\psi_-(y)(x^2), -\psi_+(x^2)(y)) = \\ &y^1 \otimes (x_1, 0)\langle x_2, y^2\rangle - y^1 \otimes (0, {y^2}_1)\langle x, {y^2}_2\rangle \\ &\qquad\qquad - x^1 \otimes ({x^2}_1, 0)\langle {x^2}_2, y\rangle + x^1 \otimes (0, y_1)\langle x^2, y_2\rangle = \\ &{x_2}^1 \otimes (x_1, 0)\langle {x_2}^2, y\rangle - y^1 \otimes (0, {y^2}_1)\langle x, {y^2}_2\rangle \\ &\qquad\qquad - x^1 \otimes ({x^2}_1, 0)\langle {x^2}_2, y\rangle + {y_2}^1 \otimes (0, y_1)\langle x, {y_2}^2\rangle\end{aligned}$$

Both expressions are equal since δ_+ and δ_- are coderivations. We leave the verification of the third condition to the reader. □

It is now very easy to see that if one applies Corollary 4.4 to the negative Yetter-Drinfel'd Lie algebra $L_- \oplus L_+$, the resulting Lie bialgebra $(L_- \oplus L_+) \oplus L_0$ is isomorphic to the Lie bialgebra $L_- \oplus L_0 \oplus L_+$ resulting from Theorem 4.3. However, it seems to be obscure in which way we can arrive at input data that satisfy the assumptions in the above Proposition. But in fact these input data can be obtained from skew-symmetric Yetter-Drinfel'd Lie algebras:

Proposition 2 Suppose that L_+ is a finite dimensional skew-symmetric positive Yetter-Drinfel'd Lie algebra over L_0. Then there is a unique negative Yetter-Drinfel'd Lie algebra structure on the dual vector space $L_- := L_+^*$ such that the evaluation

$$\langle\cdot,\cdot\rangle : L_- \times L_+ \to K, (f, y) \mapsto f(y)$$

satisfies:

1. $\langle x, \phi_+(h)(y)\rangle = -\langle\phi_-(h)(x), y\rangle$
2. $\langle x_1, y\rangle\langle x_2, y'\rangle = \langle x, [y, y']\rangle$
3. $\langle x, y_1\rangle\langle x', y_2\rangle = \langle [x, x'], y\rangle$
4. $x^1\langle x^2, y\rangle = y^1\langle x, y^2\rangle$

for all $x, x' \in L_-$, $y, y' \in L_+$ and $h \in L_0$. L_- is itself skew-symmetric and we have for $x \in L_-$, $y \in L_+$: $\phi_-(y^1)(x) \otimes y^2 = x^2 \otimes \phi_+(x^1)(y)$.

Proof. This follows essentially from Proposition 2.8. There we have constructed a positive coaction on the dual vector space that satisfies an equation which differs by a sign from the above condition 4. Changing the sign of this positive coaction, we arrive at a negative coaction that satisfies condition 4 above, and in our special case condition 2.7.3 still holds since by skew-symmetry, the right hand side is zero. We leave the proof of the few remaining assertions to the reader. $\square$

4.8 It is a natural question whether the Lie bialgebra that results from the second construction is quasitriangular. This sometimes happens if the initial Lie bialgebra L_0 was quasitriangular, namely if the coactions are related to the actions a in Subsection 2.11:

Corollary Suppose that L_0 is a quasitriangular Lie bialgebra with classical r-matrix $r = \sum_{i=1}^n a_i \otimes b_i$. Suppose that L_+ and L_- are Lie algebras which are simultaneously Lie coalgebras. Assume that

$$\phi_+ : L_0 \to Der(L_+) \quad \phi_- : L_0 \to Der(L_-)$$

are Lie algebra homomorphisms and that

$$\langle\cdot,\cdot\rangle : L_- \times L_+ \to K$$

is a bilinear form. We assume that the coproducts of L_+ and L_- are L_0-equivariant and that the following compatibility conditions are satisfied:

1. For all $y, y' \in L_+$ we have:

$$\begin{aligned}
&\Delta([y,y']) - y.\Delta(y') + y'.\Delta(y) = \\
&\qquad \sum_i \phi_+(a_i)(y) \otimes \phi_+(b_i)(y') - \sum_i \phi_+(a_i)(y') \otimes \phi_+(b_i)(y) \\
&\qquad + \sum_i \phi_+(b_i)(y) \otimes \phi_+(a_i)(y') - \sum_i \phi_+(b_i)(y') \otimes \phi_+(a_i)(y)
\end{aligned}$$

2. For all $x, x' \in L_-$ we have:

$$\begin{aligned}
&\Delta([x,x']) - x.\Delta(x') + x'.\Delta(x) = \\
&\qquad \sum_i \phi_-(a_i)(x) \otimes \phi_-(b_i)(x') - \sum_i \phi_-(a_i)(x') \otimes \phi_-(b_i)(x) \\
&\qquad + \sum_i \phi_-(b_i)(x) \otimes \phi_-(a_i)(x') - \sum_i \phi_-(b_i)(x') \otimes \phi_-(a_i)(x)
\end{aligned}$$

3. For all $x \in L_-, h \in L_0$ and $y \in L_+$ we have: $\langle x, \phi_+(h)(y)\rangle = -\langle \phi_-(h)(x), y\rangle$
4. For all $x \in L_-$ and $y, y' \in L_+$ we have: $\langle x_1, y\rangle\langle x_2, y'\rangle = \langle x, [y, y']\rangle$
5. For all $x, x' \in L_-$ and $y \in L_+$ we have: $\langle x, y_1\rangle\langle x', y_2\rangle = \langle [x, x'], y\rangle$

Define coactions

$$\delta_- : L_- \to L_0 \otimes L_-, x \mapsto \sum_i a_i \otimes \phi_-(b_i)(x)$$

$$\delta_+ : L_+ \to L_0 \otimes L_+, y \mapsto \sum_i b_i \otimes \phi_+(a_i)(y)$$

Define:

$$\{\cdot,\cdot\} : L_- \times L_+ \to L_0,\ (x,y) \mapsto \{x,y\} := y^1\langle x, y^2\rangle - x^1\langle x^2, y\rangle$$

Introduce the coadjoint actions by

$$\psi_+ : L_- \to End(L_+), \quad x \mapsto (y \mapsto \langle x, y_2\rangle y_1)$$

$$\psi_- : L_+ \to End(L_-), \quad y \mapsto (x \mapsto \langle x_2, y\rangle x_1)$$

1. Then $L_- \oplus L_0 \oplus L_+$ becomes a Lie bialgebra with the Lie bracket:

$$\begin{aligned}
&[(x,h,y),(x',h',y')] = \\
&([x,x'] + (\phi_-(h)(x') - \phi_-(h')(x)) + (\psi_-(y)(x') - \psi_-(y')(x)), \\
&[h,h'] + (\{x',y\} - \{x,y'\}), \\
&[y,y'] + (\phi_+(h)(y') - \phi_+(h')(y)) + (\psi_+(x)(y') - \psi_+(x')(y)))
\end{aligned}$$

and the two-sided cosemidirect cobracket:

$$
\begin{aligned}
&\Delta((x,h,y)) = \\
&\quad (x_1,0,0)\otimes(x_2,0,0) - (0,x^1,0)\otimes(x^2,0,0) + (x^2,0,0)\otimes(0,x^1,0) \\
&\qquad + (0,h_1,0)\otimes(0,h_2,0) \\
&\qquad - (0,0,y_1)\otimes(0,0,y_2) + (0,y^1,0)\otimes(0,0,y^2) - (0,0,y^2)\otimes(0,y^1,0)
\end{aligned}
$$

2. If L_+ and L_- are finite dimensional and the bilinear form $\langle\cdot,\cdot\rangle$ is nondegenerate, then $L_-\oplus L_0\oplus L_+$ is quasitriangular with classical r-matrix

$$R := \sum_{i=1}^{n}(0,a_i,0)\otimes(0,b_i,0) + \sum_{j=1}^{m}(x_j,0,0)\otimes(0,0,y_j)$$

where we have chosen bases $x_1,\ldots,x_m \in L_-$ and $y_1,\ldots,y_m \in L_+$ that satisfy $\langle x_i, y_j\rangle = \delta_{ij}$.

Proof. The first assertion follows from Proposition 2.11 and Theorem 4.3. To prove the second assertion, let us denote by R' the term $\sum_{i=1}^{n}(0,a_i,0)\otimes(0,b_i,0)$ and by R'' the term $\sum_{j=1}^{m}(x_j,0,0)\otimes(0,0,y_j)$. By Proposition 2.9.2, we then have:

$$
\begin{aligned}
&(x,0,0).R'' = \\
&\quad \sum_{j=1}^{m}([x,x_j],0,0)\otimes(0,0,y_j) \\
&\qquad + \sum_{j=1}^{m}(x_j,0,0)\otimes(-\psi_-(y_j)(x), -\{x,y_j\}, \psi_+(x)(y_j)) \\
&= -\sum_{j=1}^{m}(x_j,0,0)\otimes\langle x_2,y_j\rangle(x_1,0,0) - \sum_{j=1}^{m}(x_j,0,0)\otimes\langle x,{y_j}^2\rangle(0,{y_j}^1,0) \\
&\qquad + \sum_{j=1}^{m}(x_j,0,0)\otimes\langle x^2,y_j\rangle(0,x^1,0) \\
&= -(x_2,0,0)\otimes(x_1,0,0) - \sum_{i,j}(x_j,0,0)\otimes\langle x,\phi_+(a_i)(y_j)\rangle(0,b_i,0) \\
&\qquad + (x^2,0,0)\otimes(0,x^1,0)
\end{aligned}
$$

On the other hand, we have:

$$
\begin{aligned}
(x,0,0).R' &= -\sum_{i=1}^{n}(\phi_-(a_i)(x),0,0)\otimes(0,b_i,0) - \sum_{i=1}^{n}(0,a_i,0)\otimes(\phi_-(b_i)(x),0,0) \\
&= -\sum_{i=1}^{n}(\phi_-(a_i)(x),0,0)\otimes(0,b_i,0) - (0,x^1,0)\otimes(x^2,0,0)
\end{aligned}
$$

We see that the second term of the first expression and the first term of the second expression cancel and we are left with:

$$(x,0,0).R = -(x_2,0,0)\otimes(x_1,0,0) + (x^2,0,0)\otimes(0,x^1,0) - (0,x^1,0)\otimes(x^2,0,0)$$

By a similar calculation, one can show that

$$(0,0,y).R = -(0,0,y_1)\otimes(0,0,y_2) + (0,y^1,0)\otimes(0,0,y^2) - (0,0,y^2)\otimes(0,y^1,0)$$

Finally, observe that it follows from Proposition 2.9.2 that we have $(0,h,0).R'' = 0$ and therefore we can conclude from the assumption that L_0 was quasitriangular that we have $(0,h,0).R = (0,h,0).R' = (0,h_1,0)\otimes(0,h_2,0)$.

It remains to show that R satisfies the classical Yang-Baxter equation. We have:

$$\begin{aligned}
[R_{12}, R_{13}] = &\sum_{i,k}(0,[a_i,a_k],0)\otimes(0,b_i,0)\otimes(0,b_k,0)\\
&+\sum_{i,l}(\phi_-(a_i)(x_l),0,0)\otimes(0,b_i,0)\otimes(0,0,y_l)\\
&+\sum_{j,k}(-\phi_-(a_k)(x_j),0,0)\otimes(0,0,y_j)\otimes(0,b_k,0)\\
&+\sum_{j,l}([x_j,x_l],0,0)\otimes(0,0,y_j)\otimes(0,0,y_l)
\end{aligned}$$

$$\begin{aligned}
[R_{12}, R_{23}] = &\sum_{i,k}(0,a_i,0)\otimes(0,[b_i,a_k],0)\otimes(0,b_k,0)\\
&+\sum_{i,l}(0,a_i,0)\otimes(\phi_-(b_i)(x_l),0,0)\otimes(0,0,y_l)\\
&+\sum_{j,k}(x_j,0,0)\otimes(0,0,-\phi_+(a_k)(y_j))\otimes(0,b_k,0)\\
&+\sum_{j,l}(x_j,0,0)\otimes(\psi_-(y_j)(x_l),\{x_l,y_j\},-\psi_+(x_l)(y_j))\otimes(0,0,y_l)
\end{aligned}$$

and finally

$$\begin{aligned}
[R_{13}, R_{23}] = &\sum_{i,k}(0,a_i,0)\otimes(0,a_k,0)\otimes(0,[b_i,b_k],0)\\
&+\sum_{i,l}(0,a_i,0)\otimes(x_l,0,0)\otimes(0,0,\phi_+(b_i)(y_l))\\
&+\sum_{j,k}(x_j,0,0)\otimes(0,a_k,0)\otimes(0,0,-\phi_+(b_k)(y_j))\\
&+\sum_{j,l}(x_j,0,0)\otimes(x_l,0,0)\otimes(0,0,[y_j,y_l])
\end{aligned}$$

Summing up, we see that the third summand in $[R_{12}, R_{13}]$ and the third summand in $[R_{12}, R_{23}]$ cancel by Proposition 2.9.2. Likewise, the fourth summand in $[R_{12}, R_{13}]$ and the last component of the fourth summand in $[R_{12}, R_{23}]$, the fourth summand in $[R_{13}, R_{23}]$ and the first component in the fourth summand of $[R_{12}, R_{23}]$ as well as the second summand in $[R_{12}, R_{23}]$ and the second summand in $[R_{13}, R_{23}]$ cancel. Since the r-matrix of L_0 also satisfies the classical Yang-Baxter-equation, we are left with:

$$
\begin{aligned}
&[R_{12}, R_{13}] + [R_{12}, R_{23}] + [R_{13}, R_{23}] = \\
&\qquad \sum_{i,l} (\phi_-(a_i)(x_l), 0, 0) \otimes (0, b_i, 0) \otimes (0, 0, y_l) \\
&\qquad\quad + \sum_{j,l} (x_j, 0, 0) \otimes (0, \{x_l, y_j\}, 0) \otimes (0, 0, y_l) \\
&\qquad\quad + \sum_{j,k} (x_j, 0, 0) \otimes (0, a_k, 0) \otimes (0, 0, -\phi_+(b_k)(y_j)) \\
&\quad = \sum_{i,l} (\phi_-(a_i)(x_l), 0, 0) \otimes (0, b_i, 0) \otimes (0, 0, y_l) \\
&\qquad\quad + \sum_{i,j,l} (x_j, 0, 0) \otimes \langle x_l, \phi_+(a_i)(y_j)\rangle (0, b_i, 0) \otimes (0, 0, y_l) \\
&\qquad\quad - \sum_{i,j,l} (x_j, 0, 0) \otimes \langle \phi_-(b_i)(x_l), y_j\rangle (0, a_i, 0) \otimes (0, 0, y_l) \\
&\qquad\quad + \sum_{j,k} (x_j, 0, 0) \otimes (0, a_k, 0) \otimes (0, 0, -\phi_+(b_k)(y_j)) = 0 \qquad \square
\end{aligned}
$$

4.9 In the preceding Subsection, we have considered the special case of the second construction that arises when the coactions are induced from actions via an r-matrix. In this Subsection, we look at the case where the actions are induced from coactions via a dual r-matrix.

Corollary 1 Suppose that L_0 is a coquasitriangular Lie bialgebra with dual r-matrix $\langle \cdot, \cdot \rangle_0$. Suppose that L_+ is a Lie algebra which is simultaneously a Lie coalgebra and a positive Lie comodule over L_0, and that L_- is a Lie algebra which is simultaneously a Lie coalgebra and a negative Lie comodule over L_0. We assume that the following conditions are satisfied:

1. δ_+ and δ_- are 1-cocycles, that is, we have
$$\delta_+([y, y']) = y^1 \otimes [y^2, y'] + y'^1 \otimes [y, y'^2].$$
$$\delta_-([x, x']) = x^1 \otimes [x^2, x'] + x'^1 \otimes [x, x'^2].$$
2. δ_+ and δ_- are coderivations.

3. We have for all $y, y' \in L_+$:

$$\begin{aligned}
\Delta_+([y,y']) - y.\Delta_+(y') + y'.\Delta_+(y) \\
&= y^2 \otimes y'^2 \langle y'^1, y^1 \rangle_0 - y'^2 \otimes y^2 \langle y'^1, y^1 \rangle_0 \\
&\quad + y^2 \otimes y'^2 \langle y^1, y'^1 \rangle_0 - y'^2 \otimes y^2 \langle y^1, y'^1 \rangle_0
\end{aligned}$$

We have for all $x, x' \in L_-$:

$$\begin{aligned}
\Delta_-([x,x']) - x.\Delta_-(x') + x'.\Delta_-(x) \\
&= x^2 \otimes x'^2 \langle x^1, x'^1 \rangle_0 - x'^2 \otimes x^2 \langle x^1, x'^1 \rangle_0 \\
&\quad + x^2 \otimes x'^2 \langle x'^1, x^1 \rangle_0 - x'^2 \otimes x^2 \langle x'^1, x^1 \rangle_0
\end{aligned}$$

Suppose that

$$\langle \cdot, \cdot \rangle : L_- \times L_+ \to K$$

is a bilinear form with the following properties:

1. $\langle x_1, y \rangle \langle x_2, y' \rangle = \langle x, [y, y'] \rangle$
2. $\langle x, y_1 \rangle \langle x', y_2 \rangle = \langle [x, x'], y \rangle$
3. $\langle h, x^1 \rangle_0 \langle x^2, y \rangle = -\langle y^1, h \rangle_0 \langle x, y^2 \rangle$

for all $x, x' \in L_-,\ y, y' \in L_+$ and $h \in L_0$. Induce actions ϕ_-, ϕ_+ from the coactions via the dual r-matrix by the formulas

$$\begin{aligned}
\phi_+ &: L_0 \to Der(L_+), h \mapsto (y \mapsto \langle y^1, h \rangle_0 y^2) \\
\phi_- &: L_0 \to Der(L_-), h \mapsto (x \mapsto \langle h, x^1 \rangle_0 x^2)
\end{aligned}$$

Then L_+ resp. L_- become positive resp. negative Yetter-Drinfel'd Lie algebras and $\langle \cdot, \cdot \rangle$ becomes a Lie algebra biform. Define:

$$\{\cdot, \cdot\} : L_- \times L_+ \to L_0,\ (x, y) \mapsto \{x, y\} := y^1 \langle x, y^2 \rangle - x^1 \langle x^2, y \rangle$$

Denote the coadjoint action corresponding to the Lie algebra biform by

$$\psi_+ : L_- \to End(L_+) \quad \text{resp.} \quad \psi_- : L_+ \to End(L_-)$$

Then $L_- \oplus L_0 \oplus L_+$ becomes a Lie bialgebra with the Lie bracket:

$$\begin{aligned}
&[(x, h, y), (x', h', y')] = \\
&([x, x'] + (\phi_-(h)(x') - \phi_-(h')(x)) + (\psi_-(y)(x') - \psi_-(y')(x)), \\
&[h, h'] + (\{x', y\} - \{x, y'\}), \\
&[y, y'] + (\phi_+(h)(y') - \phi_+(h')(y)) + (\psi_+(x)(y') - \psi_+(x')(y)))
\end{aligned}$$

and the two-sided cosemidirect cobracket:

$$\begin{aligned}
&\Delta((x, h, y)) = \\
&\quad (x_1, 0, 0) \otimes (x_2, 0, 0) - (0, x^1, 0) \otimes (x^2, 0, 0) + (x^2, 0, 0) \otimes (0, x^1, 0) \\
&\qquad + (0, h_1, 0) \otimes (0, h_2, 0) \\
&\qquad - (0, 0, y_1) \otimes (0, 0, y_2) + (0, y^1, 0) \otimes (0, 0, y^2) - (0, 0, y^2) \otimes (0, y^1, 0)
\end{aligned}$$

Proof. The fact that L_- and L_+ are Yetter-Drinfel'd Lie algebras follows from Proposition 2.12.2. We prove that $\langle \cdot, \cdot \rangle$ is a Lie algebra biform:

$$\langle \phi_-(h)(x), y\rangle = \langle h, x^1\rangle_0 \langle x^2, y\rangle = -\langle y^1, h\rangle_0 \langle x, y^2\rangle = -\langle x, \phi_+(h)(y)\rangle$$

The result now follows from Theorem 4.3 since actions and coactions are compatible in the correct way:

$$x^2 \otimes \phi_+(x^1)(y) = \langle y^1, x^1\rangle_0 x^2 \otimes y^2 = \phi_-(y^1)(x) \otimes y^2$$

□

Analogous to the preceding Subsection, we now ask whether the Lie bialgebra $L_- \oplus L_0 \oplus L_+$ obtained in the above Corollary is itself coquasitriangular. However, this will happen only in a rather special case.

Corollary 2 Suppose that L_+, L_- and L_0 are Lie bialgebras. Assume further that L_0 is cotriangular with dual r-matrix $\langle \cdot, \cdot \rangle_0$. Suppose that L_+ is simultaneously a Lie coalgebra and a positive Lie comodule over L_0, and that L_- is simultaneously a Lie coalgebra and a negative Lie comodule over L_0 such that the coactions of L_+ and L_- are coderivations and 1-cocycles. Suppose that there are bilinear forms

$$\langle \cdot, \cdot \rangle_- : L_- \times L_- \to K \qquad \langle \cdot, \cdot \rangle_+ : L_+ \times L_+ \to K$$

such that we have for all $x, x' \in L_-$ and all $y, y' \in L_+$:

1. $[x, x'] = x_1 \langle x_2, x'\rangle_- + x'_1 \langle x, x'_2\rangle_-$
2. $[y, y'] = y_1 \langle y_2, y'\rangle_+ + y'_1 \langle y, y'_2\rangle_+$
3. $x^1 \langle x^2, x'\rangle_- = -x'^1 \langle x, x'^2\rangle_-$
4. $y^1 \langle y^2, y'\rangle_+ = -y'^1 \langle y, y'^2\rangle_+$

Suppose that

$$\langle \cdot, \cdot \rangle : L_- \times L_+ \to K$$

is a bilinear form with the following properties:

1. $\langle x_1, y\rangle \langle x_2, y'\rangle = \langle x, [y, y']\rangle$
2. $\langle x, y_1\rangle \langle x', y_2\rangle = \langle [x, x'], y\rangle$
3. $\langle h, x^1\rangle_0 \langle x^2, y\rangle = -\langle y^1, h\rangle_0 \langle x, y^2\rangle$

for all $x, x' \in L_-$, $y, y' \in L_+$ and $h \in L_0$. Induce actions ϕ_-, ϕ_+ from the coactions via the dual r-matrix by the formulas

$$\phi_+ : L_0 \to Der(L_+), h \mapsto (y \mapsto \langle y^1, h\rangle_0 y^2)$$
$$\phi_- : L_0 \to Der(L_-), h \mapsto (x \mapsto \langle h, x^1\rangle_0 x^2)$$

Define:

$$\{\cdot,\cdot\} : L_- \times L_+ \to L_0,\ (x,y) \mapsto \{x,y\} := y^1\langle x, y^2\rangle - x^1\langle x^2, y\rangle$$

Denote the coadjoint action corresponding to the Lie algebra biform by

$$\psi_+ : L_- \to End(L_+) \quad \text{resp.} \quad \psi_- : L_+ \to End(L_-)$$

Introduce the bilinear form

$$\begin{gathered}\langle\cdot,\cdot\rangle_r : (L_- \oplus L_0 \oplus L_+) \times (L_- \oplus L_0 \oplus L_+) \to K \\ ((x,h,y),(x',h',y')) \mapsto \langle x, x'\rangle_- + \langle x', y\rangle + \langle h, h'\rangle_0 - \langle x, y'\rangle + \langle y', y\rangle_+\end{gathered}$$

Then, if $L := L_- \oplus L_0 \oplus L_+$ is considered as a Lie bialgebra with the Lie bracket:

$$\begin{aligned}&[(x,h,y),(x',h',y')] = \\ &([x,x'] + (\phi_-(h)(x') - \phi_-(h')(x)) + (\psi_-(y)(x') - \psi_-(y')(x)), \\ &[h,h'] + (\{x',y\} - \{x,y'\}), \\ &[y,y'] + (\phi_+(h)(y') - \phi_+(h')(y)) + (\psi_+(x)(y') - \psi_+(x')(y)))\end{aligned}$$

and the two-sided cosemidirect cobracket:

$$\begin{aligned}&\Delta((x,h,y)) = \\ &\quad (x_1,0,0)\otimes(x_2,0,0) - (0,x^1,0)\otimes(x^2,0,0) + (x^2,0,0)\otimes(0,x^1,0) \\ &\qquad + (0,h_1,0)\otimes(0,h_2,0) \\ &\qquad - (0,0,y_1)\otimes(0,0,y_2) + (0,y^1,0)\otimes(0,0,y^2) - (0,0,y^2)\otimes(0,y^1,0),\end{aligned}$$

we have the equality:

$$[z,z'] = z_1\langle z_2, z'\rangle_r + z'_1\langle z, z'_2\rangle_r$$

for all $z, z' \in L$. If L_+ and L_- are coquasitriangular with dual r-matrices $\langle\cdot,\cdot\rangle_+$ resp. $\langle\cdot,\cdot\rangle_-$, then L is coquasitriangular with dual r-matrix $\langle\cdot,\cdot\rangle_r$.

Proof. Note first that L_+ and L_- are Yetter-Drinfel'd Lie algebras because they are skew-symmetric Yetter-Drinfel'd Lie modules. We have:

$$\begin{aligned}&(x,h,y)_1\langle (x,h,y)_2, (x',h',y')\rangle_r = \\ &\quad (x_1,0,0)\langle x_2, x'\rangle_- - (x_1,0,0)\langle x_2, y'\rangle - (0,x^1,0)\langle x^2, x'\rangle_- + (0,x^1,0)\langle x^2, y'\rangle \\ &\qquad + (x^2,0,0)\langle x^1, h'\rangle_0 + h_1\langle h_2, h'\rangle_0 - (0,0,y_1)\langle y', y_2\rangle_+ - (0,0,y_1)\langle x', y_2\rangle \\ &\qquad + (0,y^1,0)\langle y', y^2\rangle_+ + (0,y^1,0)\langle x', y^2\rangle - (0,0,y^2)\langle y^1, h'\rangle_0 \\ &\quad = (x_1\langle x_2, x'\rangle_- - \psi_-(y')(x) - \phi_-(h')(x), \\ &\qquad - x^1\langle x^2, x'\rangle_- + x^1\langle x^2, y'\rangle + h_1\langle h_2, h'\rangle_0 + y^1\langle y', y^2\rangle_+ + y^1\langle x', y^2\rangle, \\ &\qquad - y_1\langle y', y_2\rangle_+ - \psi_+(x')(y) - \phi_+(h')(y))\end{aligned}$$

Furthermore, we have:

$$
\begin{aligned}
&(x',h',y')_1\langle (x,h,y),(x',h',y')_2\rangle_r \\
&\quad = (x'_1,0,0)\langle x,x'_2\rangle_- + (x'_1,0,0)\langle x'_2,y\rangle - (0,x'^1,0)\langle x,x'^2\rangle_- \\
&\qquad - (0,x'^1,0)\langle x'^2,y\rangle + (x'^2,0,0)\langle h,x'^1\rangle_0 + h'_1\langle h,h'_2\rangle_0 \\
&\qquad - (0,0,y'_1)\langle y'_2,y\rangle_+ + (0,0,y'_1)\langle x,y'_2\rangle + (0,y'^1,0)\langle y'^2,y\rangle_+ \\
&\qquad - (0,y'^1,0)\langle x,y'^2\rangle - (0,0,y'^2)\langle h,y'^1\rangle_0 \\
&\quad = (x'_1\langle x,x'_2\rangle_- + \psi_-(y)(x') + \phi_-(h)(x'), \\
&\qquad - x'^1\langle x,x'^2\rangle_- - x'^1\langle x'^2,y\rangle + h'_1\langle h,h'_2\rangle_0 + y'^1\langle y'^2,y\rangle_+ - y'^1\langle x,y'^2\rangle, \\
&\qquad - y'_1\langle y'_2,y\rangle_+ + \psi_+(x)(y') + \phi_+(h)(y'))
\end{aligned}
$$

Forming the sum of both expressions, we get:

$$
\begin{aligned}
&(x,h,y)_1\langle (x,h,y)_2,(x',h',y')\rangle_r + (x',h',y')_1\langle (x,h,y),(x',h',y')_2\rangle_r = \\
&([x,x'] + (\phi_-(h)(x') - \phi_-(h')(x)) + (\psi_-(y)(x') - \psi_-(y')(x)), \\
&[h,h'] + (\{x',y\} - \{x,y'\}), \\
&[y,y'] + (\phi_+(h)(y') - \phi_+(h')(y)) + (\psi_+(x)(y') - \psi_+(x')(y))) \\
&\ = [(x,h,y),(x',h',y')]
\end{aligned}
$$

It remains to prove that $\langle\cdot,\cdot\rangle_r$ satisfies the dual Yang-Baxter equation if L_+ and L_- are coquasitriangular. We have:

$$
\begin{aligned}
&\langle (x,h,y)_1,(x',h',y')\rangle_r\langle (x,h,y)_2,(x'',h'',y'')\rangle_r = \\
&\ \langle x_1,x'\rangle_-\langle x_2,x''\rangle_- - \langle x_1,x'\rangle_-\langle x_2,y''\rangle - \langle x_1,y'\rangle\langle x_2,x''\rangle_- + \langle x_1,y'\rangle\langle x_2,y''\rangle \\
&\ - \langle x^1,h'\rangle_0\langle x^2,x''\rangle_- + \langle x^1,h'\rangle_0\langle x^2,y''\rangle + \langle x^2,x'\rangle_-\langle x^1,h''\rangle_0 - \langle x^2,y'\rangle\langle x^1,h''\rangle_0 \\
&\ + \langle h_1,h'\rangle_0\langle h_2,h''\rangle_0 - \langle y',y_1\rangle_+\langle y'',y_2\rangle_+ - \langle y',y_1\rangle_+\langle x'',y_2\rangle - \langle x',y_1\rangle\langle y'',y_2\rangle_+ \\
&\ - \langle x',y_1\rangle\langle x'',y_2\rangle + \langle y^1,h'\rangle_0\langle x'',y^2\rangle + \langle y^1,h'\rangle_0\langle y'',y^2\rangle_+ - \langle x',y^2\rangle\langle y^1,h''\rangle_0 \\
&\ - \langle y',y^2\rangle_+\langle y^1,h''\rangle_0
\end{aligned}
$$

The next term occurring in the dual Yang-Baxter equation is:

$$
\begin{aligned}
&\langle (x,h,y),(x',h',y')_1\rangle_r\langle (x',h',y')_2,(x'',h'',y'')\rangle_r = \\
&\ \langle x,x'_1\rangle_-\langle x'_2,x''\rangle_- - \langle x,x'_1\rangle_-\langle x'_2,y''\rangle + \langle x'_1,y\rangle\langle x'_2,x''\rangle_- - \langle x'_1,y\rangle\langle x'_2,y''\rangle \\
&\ - \langle h,x'^1\rangle_0\langle x'^2,x''\rangle_- + \langle h,x'^1\rangle_0\langle x'^2,y''\rangle + \langle x,x'^2\rangle_-\langle x'^1,h''\rangle_0 \\
&\ + \langle x'^2,y\rangle\langle x'^1,h''\rangle_0 + \langle h,h'_1\rangle_0\langle h'_2,h''\rangle_0 - \langle y'_1,y\rangle_+\langle y'',y'_2\rangle_+ - \langle y'_1,y\rangle_+\langle x'',y'_2\rangle \\
&\ + \langle x,y'_1\rangle\langle y'',y'_2\rangle_+ + \langle x,y'_1\rangle\langle x'',y'_2\rangle + \langle h,y'^1\rangle_0\langle y'',y'^2\rangle_+ + \langle h,y'_1\rangle_0\langle x'',y'^2\rangle \\
&\ - \langle y'^2,y\rangle_+\langle y'^1,h''\rangle_0 + \langle x,y'^2\rangle\langle y'^1,h''\rangle_0
\end{aligned}
$$

Finally, the last term in the dual Yang-Baxter equation is:

$$
\begin{aligned}
&\langle (x,h,y),(x'',h'',y'')_1\rangle_r \langle (x',h',y'),(x'',h'',y'')_2\rangle_r = \\
&\quad \langle x,x''_1\rangle_- \langle x',x''_2\rangle_- + \langle x''_1,y\rangle_- \langle x''_2,y'\rangle + \langle x''_1,y\rangle\langle x',x''_2\rangle_- + \langle x,x''_1\rangle_- \langle x''_2,y'\rangle \\
&\quad - \langle h,x''^1\rangle_0 \langle x',x''^2\rangle_- - \langle h,x''^1\rangle_0 \langle x''^2,y'\rangle + \langle x,x''^2\rangle_- \langle h',x''^1\rangle_0 \\
&\quad + \langle x''^2,y\rangle\langle h',x''^1\rangle_0 + \langle h,h''_1\rangle_0 \langle h',h''_2\rangle_0 - \langle y''_1,y\rangle_+ \langle y''_2,y'\rangle + \langle y''_1,y\rangle_+ \langle x',y''_2\rangle \\
&\quad + \langle x,y''_1\rangle\langle y''_2,y'\rangle_+ - \langle x,y''_1\rangle\langle x',y''_2\rangle + \langle h,y''^1\rangle_0 \langle y''^2,y'\rangle_+ - \langle h,y''^1\rangle_0 \langle x',y''^2\rangle \\
&\quad - \langle y''^2,y\rangle_+ \langle h',y''^1\rangle_0 + \langle x,y''^2\rangle\langle h',y''^1\rangle_0
\end{aligned}
$$

Forming the sum of these expressions, we see that three times three terms vanish because $\langle\cdot,\cdot\rangle_+$, $\langle\cdot,\cdot\rangle_-$ and $\langle\cdot,\cdot\rangle_0$ satisfy the dual Yang-Baxter equation. Using the skew-symmetry of the dual r-matrix of L_0, we see by inspection that all terms that contain $\langle\cdot,\cdot\rangle_0$ vanish. We are therefore left with:

$$
\begin{aligned}
&\langle (x,h,y)_1,(x',h',y')\rangle_r \langle (x,h,y)_2,(x'',h'',y'')\rangle_r + \\
&\langle (x,h,y),(x',h',y')_1\rangle_r \langle (x',h',y')_2,(x'',h'',y'')\rangle_r + \\
&\langle (x,h,y),(x'',h'',y'')_1\rangle_r \langle (x',h',y'),(x'',h'',y'')_2\rangle_r = \\
&\quad - \langle x_1,x'\rangle_- \langle x_2,y''\rangle - \langle x_1,y'\rangle\langle x_2,x''\rangle_- + \langle x,[y',y'']\rangle - \langle y',y_1\rangle_+ \langle x'',y_2\rangle \\
&\quad - \langle x',y_1\rangle\langle y'',y_2\rangle_+ - \langle [x',x''],y\rangle - \langle x,x'_1\rangle_- \langle x'_2,y''\rangle + \langle x'_1,y\rangle\langle x'_2,x''\rangle_- \\
&\quad - \langle x',[y,y'']\rangle - \langle y'_1,y\rangle_+ \langle x'',y'_2\rangle + \langle x,y'_1\rangle\langle y'',y'_2\rangle_+ + \langle [x,x''],y'\rangle \\
&\quad + \langle x,x''_1\rangle_- \langle x''_2,y'\rangle + \langle x''_1,y\rangle\langle x',x''_2\rangle_- + \langle x'',[y,y']\rangle + \langle y''_1,y\rangle_+ \langle x',y''_2\rangle \\
&\quad + \langle x,y''_1\rangle\langle y''_2,y'\rangle_+ - \langle [x,x'],y''\rangle
\end{aligned}
$$

But this expression vanishes since we have assumed that the Lie bracket of L_+, L_0 resp. L_0 is linked to the cobracket via the dual r-matrix. $\square$

Note that this case of the second construction is intimately linked to the skew-symmetric case considered in Subsection 4.7, because in most cases our hypothesis $\langle h,x^1\rangle_0\langle x^2,y\rangle = -\langle y^1,h\rangle_0\langle x,y^2\rangle$ will be satisfied because the dual r-matrix is skew-symmetric and the equation $x^1\langle x^2,y\rangle = y^1\langle x,y^2\rangle$ holds.

4.10 Next we consider the question which Lie algebras can be constructed by means of the second construction. In contrast to the first construction, we shall here only give a sufficient condition: If the Lie algebra admits a triangular decomposition and possesses a bilinear form which is, as the Killing form, compatible with the triangular decomposition, then it has the form of the second construction.

Theorem Suppose that L is a finite dimensional Lie algebra over a field K whose characteristic is different from two. Suppose that

$$L = L_- \oplus L_0 \oplus L_+$$

is a decomposition into three Lie subalgebras. Let

$$\langle\cdot,\cdot\rangle_K : L \times L \to K$$

be a nondegenerate, symmetric, invariant bilinear form with the following properties:

1. L_- and L_+ are totally isotropic, that is, the restriction of $\langle \cdot, \cdot \rangle_K$ to these subalgebras is zero.
2. L_0 is orthogonal to L_- and L_+.
3. For all $h \in L_0$ and $x \in L_-$, we have $[h, x] \in L_-$.
4. For all $h \in L_0$ and $y \in L_+$, we have $[h, y] \in L_+$.

Regard L_0 as a coabelian Lie bialgebra. Then L_+ can be turned into a symmetric Yetter-Drinfel'd Lie algebra in such a way that the Lie algebra $L_+^* \oplus L_0 \oplus L_+$ resulting from the second construction is isomorphic to L.

Proof. It follows directly from the assumptions that the restriction of $\langle \cdot, \cdot \rangle_K$ to $L_0 \times L_0$ is nondegenerate. Similarly, the bilinear form induces a nondegenerate pairing if it is restricted to $L_- \times L_+$. We also know from Theorem 3.3 that there are Lie algebra homomorphisms

$$\begin{array}{ll} \phi_+ : L_0 \to Der(L_+) & \phi_- : L_0 \to Der(L_-) \\ \psi_+ : L_- \to End(L_+) & \psi_- : L_+ \to End(L_-) \end{array}$$

and a bilinear map:

$$\{\cdot, \cdot\} : L_- \times L_+ \to L_0, (x, y) \mapsto \{x, y\}$$

such that the Lie bracket of L has the form stated in that Theorem. Because L_- and L_+ are dual to each other with respect to $\langle \cdot, \cdot \rangle_K$, the Lie bracket on L_- induces a Lie cobracket on L_+ and vice versa. As a first step, we show that ψ_+ and ψ_- are the coadjoint actions. This follows directly from Proposition 2.9.1 and the invariance of the bilinear form:

$$\langle \psi_-(y)(x), y' \rangle = \langle [y, x], y' \rangle = -\langle x, [y, y'] \rangle$$

and similarly for ψ_+. Two of the axioms for a Lie algebra biform already hold by definition of the coproducts. The last one follows by virtually the same calculation as above:

$$\langle x, \phi_+(h)(y) \rangle = \langle x, [h, y] \rangle = -\langle [h, x], y \rangle = -\langle \phi_-(h)(x), y \rangle$$

Two things remain to be done: First, we have to define linear maps

$$\delta_+ : L_+ \to L_0 \otimes L_+ \qquad \delta_- : L_- \to L_0 \otimes L_-$$

that turn L_+ and L_- into Yetter-Drinfel'd Lie algebras, and second, we have to identify the last structure element in the equality:

$$\{x, y\} = y^1 \langle x, y^2 \rangle - x^1 \langle x^2, y \rangle$$

We introduce the bilinear form

$$\langle\cdot,\cdot\rangle_0 : L_0 \times L_0 \to K, (h,h') \mapsto 2\,\langle h,h'\rangle_K$$

which is just twice the original form, and select dual bases $a_1,\ldots,a_n$ and $b_1,\ldots,b_n$ satisfying $\langle a_i,b_j\rangle_0 = \delta_{ij}$. As in Proposition 2.9.2, it follows from the invariance of the bilinear form that the element $r = \sum_{i=1}^n a_i \otimes b_i$ is also invariant, and therefore, since we have considered L_0 as coabelian, L_0 is a coboundary Lie bialgebra whose cobracket is the coboundary of r. We now carry out the first step of our program by defining:

$$\delta_+(y) = \sum_{i=1}^n a_i \otimes \phi_+(b_i)(y) \qquad \delta_-(x) = \sum_{i=1}^n a_i \otimes \phi_-(b_i)(x)$$

for $y \in L_+$ and $x \in L_-$. With this choice, our Lie algebra biform is even strict:

$$x^1\langle x^2,y\rangle = \sum_{i=1}^n a_i\langle\phi_-(b_i)(x),y\rangle = -\sum_{i=1}^n a_i\langle x,\phi_+(b_i)(y)\rangle = -y^1\langle x,y^2\rangle$$

This implies by Proposition 2.8, that the structures on L_- are precisely those induced on the dual vector space of L_+. It is also easy to see that L_+ is symmetric. Before completing the first step of our program it is useful to carry out the second step. We have:

$$\begin{aligned}\langle\{x,y\},h\rangle_K &= \langle[y,x],h\rangle_K = -\langle x,[y,h]\rangle_K = \langle x,\phi_+(h)(y)\rangle_K \\ &= \sum_{i=1}^n \langle h,a_i\rangle_0\langle x,\phi_+(b_i)(y)\rangle_K = \langle h,y^1\rangle_0\langle x,y^2\rangle_K = 2\,\langle h,y^1\rangle_K\langle x,y^1\rangle_K\end{aligned}$$

and since our form is nondegenerate we conclude:

$$\{x,y\} = 2\,y^1\langle x,y^2\rangle_K = y^1\langle x,y^2\rangle_K - x^1\langle x^2,y\rangle_K$$

What we want now is to conclude by Proposition 2.11 that the structures introduced so far turn L_+ into a Yetter-Drinfel'd Lie algebra. This means that we have to verify the compatibility conditions in that Proposition. The equivariance of the coproduct of L_+ follows from dualizing the fact that $\phi_-(h)$ restricts to a derivation of L_-. The second compatibility condition follows from a calculation similar to the one we have used in the second step in the proof of Theorem 4.2:

$$\begin{aligned}&\langle x'\otimes x,\Delta_+([y,y']) - y.\Delta_+(y') + y'.\Delta(y)\rangle_K \\ &=\langle x',\psi_+(x)([y,y']) - [\psi_+(x)(y),y'] + [y,\psi_+(x)(y')] \\ &\qquad - \psi_+(\psi_-(y')(x))(y) - \psi_+(\psi_-(y')(x))(y')\rangle_K \\ &=\langle x',\phi_+(\{x,y'\})(y) - \phi_+(\{x,y\})(y')\rangle_K \\ &=2\,\sum_{i=1}^n\langle x'\otimes x,\phi_+(b_i)(y)\otimes\phi_+(a_i)(y') - \phi_+(a_i)(y')\otimes\phi_+(b_i)(y)\rangle_K\end{aligned}$$

Now we can conclude from Proposition 2.11 that we have turned L_+ and L_- into Yetter-Drinfel'd Lie algebras. We now want to apply the second construction in the form of Theorem 4.2. For this, we have to verify the hypotheses of this Theorem. But this has essentially already been done: Observing that our classical r-matrix is symmetric, the compatibility between δ_+ and ϕ_+ (resp. δ_- and ϕ_-) can be shown as in the proof of Proposition 2.4 and the equation $x^2 \otimes \phi_+(x^1)(y) = \phi_-(y^1)(x) \otimes y^2$ can be shown as in the proof of Corollary 4.6. We therefore obtain by Theorem 4.2 a Lie algebra structure on $L_- \oplus L_0 \oplus L_+$ which is equal to the original Lie algebra structure because we already have identified the structure elements $\phi_+, \phi_-, \psi_+, \psi_-$ and $\{\cdot, \cdot\}$. $\square$

5 Kac-Moody algebras

5.1 We shall encounter some special cases of concepts from the cohomology theory of Lie algebras, as we already did in the previous Sections. In order to improve the understanding of these special cases, we note here how they arise from the general case. The reader should refer to the textbooks [1], [7] or [19] for details on Lie algebra cohomology, as well as to the original work [3].

If L is a Lie algebra and V is an L-module, the cohomology groups of L with values in V are the cohomology groups of the Chevalley-Eilenberg complex whose n-th cochain group is $Hom_K(\bigwedge^n L, V)$ and whose coboundary operator

$$d^n : Hom_K(\bigwedge^n L, V) \to Hom_K(\bigwedge^{n+1} L, V)$$

is given by the formula

$$(d^n f)(\bigwedge_{i=1}^{n+1} x_i) = \sum_{i=1}^{n+1} (-1)^{i-1} x_i . f(\bigwedge_{\substack{j=1 \\ j \neq i}}^{n+1} x_j)$$
$$+ \sum_{\substack{i,j=1 \\ i<j}}^{n+1} (-1)^{i+j} f([x_i, x_j] \wedge \bigwedge_{\substack{k=1 \\ k \neq i,j}}^{n+1} x_k)$$

In particular, a 1-cocycle is a map $f : L \to V$ that satisfies $f([x, y]) = x.f(y) - y.f(x)$ for all $x, y \in L$. A 2-cocycle is a map $f : L \wedge L \to V$ that satisfies $x.f(y \wedge z) + y.f(z \wedge x) + z.f(x \wedge y) = f([x, y] \wedge z) + f([y, z] \wedge x) + f([z, x] \wedge y)$ for all $x, y, z \in L$. Note that an equivariant map $g : V \to W$ between two L-modules induces a chain map g_* between the corresponding Chevalley-Eilenberg complexes:

$$g_* : Hom(\bigwedge^n L, V) \to Hom(\bigwedge^n L, W), f \mapsto g \circ f$$

We shall need two facts from Lie algebra cohomology:

Proposition 1 Suppose that L is a free Lie algebra over the generators $X \subset L$ and that V is an L-module.

1. The cohomology groups $H^n(L,V)$ vanish for $n \geq 2$.
2. Suppose that for every $x \in X$ we are given some element $v_x \in V$. Then there is a unique 1-cocycle $f : L \to V$ satisfying $f(x) = v_x$ for all $x \in X$.

Proof. See [19], Corollary 7.2.5, p. 222, or [7], Chap. 7, Corollary 2.6, p.237. See also [14], Lemma 0.7 iii), p. 8 for proof in a special case. □

For the second fact, we need a short preparation. Obviously, L is in a natural way a module over the Lie algebra $Der(L)$. Therefore, $\bigotimes^n L$ and $\bigwedge^n L$ are $Der(L)$-modules, and therefore also $Hom(\bigwedge^n L, \bigotimes^m L)$. In this situation, the following Proposition holds:

Proposition 2 The coboundary operator of the Chevalley-Eilenberg-complex $d^n : Hom_K(\bigwedge^n L, \bigotimes^m L) \to Hom_K(\bigwedge^{n+1} L, \bigotimes^m L)$ is $Der(L)$-equivariant.

Proof. This can be shown by direct calculation using the formula

$$D.(x.y) = (D(x)).y + x.(D.y)$$

for $D \in Der(L)$, $x \in L$ and $y \in \bigotimes^m L$. □

5.2 We have already noted in Subsection 2.6 that under certain conditions a certain mapping is equivariant. Under some other conditions, the same map is a 2-cocycle:

Proposition Suppose that L_0 and L are two Lie algebras, and suppose that $\phi : L_0 \to Der(L)$ is a Lie algebra homomorphism. Suppose that $\delta : L \to L_0 \otimes L$ is a 1-cocycle in the sense that we have

$$\delta([y, y']) = y^1 \otimes [y^2, y'] + y'^1 \otimes [y, y'^2]$$

for all $y, y' \in L$. Then the mapping

$$f : L \wedge L \to L \otimes L$$
$$y \wedge y' \mapsto y^2 \otimes \phi(y^1)(y') - \phi(y^1)(y') \otimes y^2 + \phi(y'^1)(y) \otimes y'^2 - y'^2 \otimes \phi(y'^1)(y)$$

is a 2-cocycle.

Proof. We have:

$$\begin{aligned} y.f(y' \wedge y'') = \\ & [y, y'^2] \otimes \phi(y'^1)(y'') + y'^2 \otimes [y, \phi(y'^1)(y'')] \\ & - [y, \phi(y'^1)(y'')] \otimes y'^2 - \phi(y'^1)(y'') \otimes [y, y'^2] \\ & + [y, \phi(y''^1)(y')] \otimes y''^2 + \phi(y''^1)(y') \otimes [y, y''^2] \\ & - [y, y''^2] \otimes \phi(y''^1)(y') - y''^2 \otimes [y, \phi(y''^1)(y')] \end{aligned}$$

On the other hand, we have:

$$
\begin{aligned}
&f([y,y']\wedge y'') = \\
&\qquad [y,y']^2 \otimes \phi([y,y']^1)(y'') - \phi([y,y']^1)(y'') \otimes [y,y']^2 \\
&\qquad + \phi(y''^1)([y,y']) \otimes y''^2 - y''^2 \otimes \phi(y''^1)([y,y']) = \\
&\qquad [y^2,y'] \otimes \phi(y^1)(y'') + [y,y'^2] \otimes \phi(y'^1)(y'') \\
&\qquad - \phi(y^1)(y'') \otimes [y^2,y'] - \phi(y'^1)(y'') \otimes [y,y'^2] \\
&\qquad + [\phi(y''^1)(y),y'] \otimes y''^2 + [y,\phi(y''^1)(y')] \otimes y''^2 \\
&\qquad - y''^2 \otimes [\phi(y''^1)(y),y'] - y''^2 \otimes [y,\phi(y''^1)(y')]
\end{aligned}
$$

We have to prove that the cyclic sums over these expressions are equal. At first, we see that the first term of the first expression equals the second term of the second expression. Sometimes only up to cyclic permutations, we see also that the second term of the first expression equals the seventh term of the second expression, the third term equals the fifth, the fourth term equals the fourth, the fifth equals the sixth, the sixth equals the third, the seventh equals the first, and finally the eighth term of the first expression equals the eighth term of the second expression. $\square$

5.3 We now begin to give a construction of symmetrizable Kac-Moody algebras based on the second construction considered in the previous Section. We shall follow closely the notations and conventions used in [9], in particular, we work over the complex numbers, although this is not really necessary. Let us nevertheless recall the basic definitions.

Definition

1. A realization of an arbitrary complex $n \times n$-matrix $A = (a_{ij})$ is a triple $(H, \Pi, \Pi^\vee)$ consisting of a complex vector space H of dimension $2n - rank(A)$ and linearly independent sets $\Pi^\vee = \{\alpha_1^\vee, \dots, \alpha_n^\vee\}$ in H and $\Pi = \{\alpha_1, \dots, \alpha_n\}$ in the dual vector space H^* such that for all $i, j = 1, \dots, n$ the equality
$$\alpha_j(\alpha_i^\vee) = a_{ij}$$
holds.

2. A complex matrix A is called symmetrizable if it can be written as
$$A = DB$$
where $D = diag(\epsilon_1, \dots, \epsilon_n)$ is an invertible diagonal matrix and $B = (b_{ij})$ is symmetric.

It can be shown that every complex matrix has a realization which is essentially unique (cf. [9], Proposition 1.1, p. 2). For the rest of this Section, we shall work with a fixed complex matrix A, a fixed symmetrization and a fixed realization.

Following [9], Section 2.1 we introduce a symmetric bilinear form $\langle\cdot,\cdot\rangle_H$ on H by choosing a fixed complement H'' of $H' := Span\{\alpha_1^\vee, \ldots, \alpha_n^\vee\}$ and defining

$$\langle\alpha_i^\vee, \alpha_j^\vee\rangle_H = b_{ij}\epsilon_i\epsilon_j \qquad \langle\alpha_i^\vee, h\rangle_H = \alpha_i(h)\epsilon_i \qquad \langle h, h'\rangle_H = 0$$

for $h, h' \in H''$. This form is nondegenerate by [9], Lemma 2.1, p. 17. We shall also need twice this form, for which we introduce its own notation:

$$\langle\cdot,\cdot\rangle_0 : H \times H \to \mathbb{C}, (h, h') \mapsto 2\langle h, h'\rangle_H$$

We consider H as an abelian and coabelian Lie bialgebra. H is then coquasitriangular with dual r-matrix $\langle\cdot,\cdot\rangle_0$, since this is true for any choice of a dual r-matrix.

5.4 Let $\tilde{L}_+$ and $\tilde{L}_-$ be two free Lie algebras with generators $\tilde{e}_1, \ldots, \tilde{e}_n$ resp. $\tilde{f}_1, \ldots, \tilde{f}_n$. By Proposition 5.1.1, there are unique 1-cocycles

$$\tilde{\delta}_+ : \tilde{L}_+ \to H \otimes \tilde{L}_+ \qquad \tilde{\delta}_- : \tilde{L}_- \to H \otimes \tilde{L}_-$$

that satisfy on the generators:

$$\tilde{\delta}_+(\tilde{e}_i) = \frac{1}{2\epsilon_i}\alpha_i^\vee \otimes \tilde{e}_i \qquad \tilde{\delta}_-(\tilde{f}_j) = -\frac{1}{2\epsilon_j}\alpha_j^\vee \otimes \tilde{f}_j$$

Here, we have considered H as a trivial module over $\tilde{L}_+$ (resp. $\tilde{L}_-$).

Proposition

1. $\tilde{\delta}_+$ is a positive coaction.

2. $\tilde{\delta}_-$ is a negative coaction.

Proof. Observe that positive and negative coactions are the same since H is coabelian. Denote by $\tau : H \otimes H \to H \otimes H$ the map that interchanges the tensorands. We prove that the map

$$\rho := (id \otimes \tilde{\delta}_+) \circ \tilde{\delta}_+ - (\tau \otimes id) \circ (id \otimes \tilde{\delta}_+) \circ \tilde{\delta}_+$$

is a 1-cocycle:

$$\begin{aligned}(id \otimes \tilde{\delta}_+)\tilde{\delta}_+([y, y']) &= (id \otimes \tilde{\delta}_+)(y^1 \otimes [y^2, y'] + y'^1 \otimes [y, y'^2]) \\ &= y^1 \otimes y^{21} \otimes [y^{22}, y'] + y^1 \otimes y'^1 \otimes [y^2, y'^2] \\ &\quad + y'^1 \otimes y^1 \otimes [y^2, y'^2] + y'^1 \otimes y'^{21} \otimes [y, y'^{22}]\end{aligned}$$

Therefore, we have $\rho([y, y']) = y.\rho(y') - y'.\rho(y)$. Since we have $\rho(\tilde{e}_i) = 0$, we can conclude from Proposition 5.1.1 that we must have $\rho = 0$. The case of $\tilde{\delta}_-$ can be treated similarly. □

By Proposition 2.5.2, we can introduce actions

$$\tilde{\phi}_+ : H \longrightarrow End(\tilde{L}_+) \qquad \tilde{\phi}_- : H \longrightarrow End(\tilde{L}_-)$$

that turn $\tilde{L}_-$ (resp. $\tilde{L}_+$) into a negative (resp. positive) Yetter-Drinfel'd Lie module via the formulas:

$$\tilde{\phi}_+(h)(y) = \langle y^1, h\rangle_0 y^2 \qquad \tilde{\phi}_-(h)(x) = \langle h, x^1\rangle_0 x^2$$

for $x \in \tilde{L}_-$, $y \in \tilde{L}_+$. Since $\tilde{\delta}_+$ and $\tilde{\delta}_-$ are 1-cocycles, we know by Proposition 2.12.1 that $\tilde{\phi}_+(h)$ and $\tilde{\phi}_-(h)$ are derivations. Since derivations are nothing but 1-cocycles, they are also determined by their values on generators. $\tilde{\phi}_+(h)$ takes the value:

$$\tilde{\phi}_+(h)(\tilde{e}_i) = \langle y^1, h\rangle_0 y^2 = \frac{1}{2\epsilon_i}\langle \alpha_i^\vee, h\rangle_0 \tilde{e}_i = \alpha_i(h)\tilde{e}_i$$

Similarly, we have $\tilde{\phi}_-(h)(\tilde{f}_j) = -\alpha_j(h)\tilde{f}_j$, and this would have been an alternative definition of the actions. Since $\langle\cdot,\cdot\rangle_0$ is symmetric, $\tilde{L}_+$ and $\tilde{L}_-$ become symmetric Yetter-Drinfel'd Lie modules. It follows from Proposition 2.6 and Proposition 5.2 that the mapping $f : \tilde{L}_+ \wedge \tilde{L}_+ \to \tilde{L}_+ \otimes \tilde{L}_+$ given by

$$f(y \wedge y') = y^2 \otimes \tilde{\phi}_+(y^1)(y') - \tilde{\phi}_+(y^1)(y') \otimes y^2 + \tilde{\phi}_+(y'^1)(y) \otimes y'^2 - y'^2 \otimes \tilde{\phi}_+(y'^1)(y)$$

is an H-equivariant 2-cocycle. We note that by symmetry, f is also given by the shorter formula:

$$f(y \wedge y') = 2\; y^2 \otimes \tilde{\phi}_+(y^1)(y') - 2\; \tilde{\phi}_+(y^1)(y') \otimes y^2$$

Since the second cohomology group of $\tilde{L}_+$ vanishes by Proposition 5.1.1, there is a mapping

$$\tilde{\Delta}_+ : \tilde{L}_+ \to \tilde{L}_+ \otimes \tilde{L}_+$$

that satisfies:

$$(d^1\tilde{\Delta}_+)(y \wedge y') = \tilde{\Delta}_+([y, y']) - y.\tilde{\Delta}_+(y') + y'.\tilde{\Delta}_+(y) = 2\; y^2 \otimes \tilde{\phi}_+(y^1)(y') - 2\; \tilde{\phi}_+(y^1)(y') \otimes y^2$$

for all $y, y' \in \tilde{L}_+$. This mapping is of course unique up to a 1-cocycle. Since, according to Proposition 5.1.1, 1-cocycles are determined by their values on generators, the mapping $\tilde{\Delta}_+$ becomes unique if we require the condition: $\tilde{\Delta}_+(\tilde{e}_i) = 0$ for $i = 1, \ldots, n$. Similarly, there is a unique mapping

$$\tilde{\Delta}_- : \tilde{L}_- \to \tilde{L}_- \otimes \tilde{L}_-$$

that vanishes on the generators $\tilde{f}_1, \ldots, \tilde{f}_n$ and whose coboundary satisfies:

$$\tilde{\Delta}_-([x, x']) - x.\tilde{\Delta}_-(x') + x'.\tilde{\Delta}_+(x) = 2\; x^2 \otimes \tilde{\phi}_-(x^1)(x') - 2\; \tilde{\phi}_-(x^1)(x') \otimes x^2$$

for all $x, x' \in \tilde{L}_-$.

Theorem $\tilde{L}_+$ and $\tilde{L}_-$ are Yetter-Drinfel'd Lie algebras over H.

Proof. We postpone the proof that $\tilde{L}_+$ and $\tilde{L}_-$ are Lie coalgebras and verify the compatibility conditions first. All of them are satisfied by construction, except for the equivariance of the coproduct and the fact that the coactions are coderivations. Since both assertions are equivalent by Proposition 2.12.1, we only verify the equivariance of the coproduct:

Consider the mapping

$$f : \tilde{L}_+ \wedge \tilde{L}_+ \to \tilde{L}_+ \otimes \tilde{L}_+, y \wedge y' \mapsto 2\, y^2 \otimes \tilde{\phi}_+(y^1)(y') - 2\, \tilde{\phi}_+(y^1)(y') \otimes y^2$$

As explained in Subsection 5.1, $Hom(\tilde{L}_+ \wedge \tilde{L}_+, \tilde{L}_+ \otimes \tilde{L}_+)$ is a $Der(\tilde{L}_+)$-module, and therefore an H-module by pullback via $\tilde{\phi}_+$. The fact that f is H-equivariant just means that f is annihilated by all $\tilde{\phi}_+(h), h \in H$. But, since $f = d^1\tilde{\Delta}_+$, and since by Proposition 5.1.2 the Chevalley-Eilenberg coboundary operator is equivariant, we have:

$$d^1(\tilde{\phi}_+(h).\tilde{\Delta}_+) = \tilde{\phi}_+(h).d^1\tilde{\Delta}_+ = \tilde{\phi}_+(h).f = 0$$

Therefore, $\tilde{\phi}_+(h).\tilde{\Delta}_+$ is a 1-cocycle. Since 1-cocycles are, by Proposition 5.1.1, determined by their values on generators, and we have

$$(\tilde{\phi}_+(h).\tilde{\Delta}_+)(\tilde{e}_i) = \tilde{\phi}_+(h).(\tilde{\Delta}_+(\tilde{e}_i)) - \tilde{\Delta}_+(\tilde{\phi}_+(h)(\tilde{e}_i)) = 0,$$

we conclude that $\tilde{\phi}_+(h).\tilde{\Delta}_+ = 0$, that is, $\tilde{\Delta}_+$ is H-equivariant.

It remains to show that $\tilde{L}_+$ is a Lie coalgebra. We first show the antisymmetry. Consider the map $\tau : \tilde{L}_+ \otimes \tilde{L}_+ \to \tilde{L}_+ \otimes \tilde{L}_+, y \otimes y' \mapsto y' \otimes y$. It is easy to see that it is equivariant and therefore induces a chain map τ_* of the Chevalley-Eilenberg complex, as explained in Subsection 5.1. We now calculate:

$$d^1(\tau_*(\tilde{\Delta}_+)) = \tau_*(d^1\tilde{\Delta}_+) = \tau_*(f) = -f = -d^1\tilde{\Delta}_+$$

Therefore $\tau \circ \tilde{\Delta}_+ + \tilde{\Delta}_+$ is a 1-cocycle. But since it vanishes on the generators $\tilde{e}_1, \ldots, \tilde{e}_n$ by construction, it must vanish identically by Proposition 5.1.1.

We now prove the co-Jacobi identity. Consider the two mappings

$$\begin{gathered} p : \tilde{L}_+ \otimes \tilde{L}_+ \otimes \tilde{L}_+ \to \tilde{L}_+ \otimes \tilde{L}_+ \otimes \tilde{L}_+ \\ y \otimes y' \otimes y'' \mapsto y \otimes y' \otimes y'' + y'' \otimes y \otimes y' + y' \otimes y'' \otimes y \end{gathered}$$

and

$$\begin{gathered} g : \tilde{L}_+ \to \tilde{L}_+ \otimes \tilde{L}_+ \otimes \tilde{L}_+ \\ y \mapsto (\tilde{\Delta}_+ \otimes id_{\tilde{L}_+})\tilde{\Delta}_+(y) \end{gathered}$$

We have to show that the composition $p \circ g$ is zero. For this purpose, we first show that it is a 1-cocycle. We have:

$$\begin{aligned}
g([y,y']) = (\tilde{\Delta}_+ \otimes id_{\tilde{L}_+})([y,y'_1] \otimes y'_2 + y'_1 \otimes [y,y'_2] - [y',y_1] \otimes y_2 - y_1 \otimes [y',y_2] \\
+ 2\, y^2 \otimes \tilde{\phi}_+(y^1)(y') - 2\, \tilde{\phi}_+(y^1)(y') \otimes y^2) \\
= [y,y'_{11}] \otimes y'_{12} \otimes y'_2 + y'_{11} \otimes [y,y'_{12}] \otimes y'_2 \\
- [y'_1,y_1] \otimes y_2 \otimes y'_2 - y_1 \otimes [y'_1,y_2] \otimes y'_2 \\
+ 2\, y^2 \otimes \tilde{\phi}_+(y^1)(y'_1) \otimes y'_2 - 2\, \tilde{\phi}_+(y^1)(y'_1) \otimes y^2 \otimes y'_2 \\
+ y'_{11} \otimes y'_{12} \otimes [y,y'_2] - [y',y_{11}] \otimes y_{12} \otimes y_2 \\
- y_{11} \otimes [y',y_{12}] \otimes y_2 + [y_1,y'_1] \otimes y'_2 \otimes y_2 \\
+ y'_1 \otimes [y_1,y'_2] \otimes y_2 - 2\, y'^2 \otimes \tilde{\phi}_+(y'^1)(y_1) \otimes y_2 \\
+ 2\, \tilde{\phi}_+(y'^1)(y_1) \otimes y'^2 \otimes y_2 - y_{11} \otimes y_{12} \otimes [y',y_2] \\
+ 2\, {y^2}_1 \otimes {y^2}_2 \otimes \tilde{\phi}_+(y^1)(y') - 2\, \tilde{\phi}_+(y^1)(y'_1) \otimes y'_2 \otimes y^2 \\
- 2\, y'_1 \otimes \tilde{\phi}_+(y^1)(y'_2) \otimes y^2
\end{aligned}$$

This implies:

$$\begin{aligned}
(d^1 g)(y \wedge y') = g([y,y']) - y.g(y') + y'.g(y) \\
= - [y'_1,y_1] \otimes y_2 \otimes y'_2 - y_1 \otimes [y'_1,y_2] \otimes y'_2 \\
+ [y_1,y'_1] \otimes y'_2 \otimes y_2 + y'_1 \otimes [y_1,y'_2] \otimes y_2 \\
+ 2\, y^2 \otimes \tilde{\phi}_+(y^1)(y'_1) \otimes y'_2 - 2\, \tilde{\phi}_+(y^1)(y'_1) \otimes y^2 \otimes y'_2 \\
- 2\, y'^2 \otimes \tilde{\phi}_+(y'^1)(y_1) \otimes y_2 + 2\, \tilde{\phi}_+(y'^1)(y_1) \otimes y'^2 \otimes y_2 \\
+ 2\, {y_1}^2 \otimes y_2 \otimes \tilde{\phi}_+({y_1}^1)(y') + 2\, y_1 \otimes {y_2}^2 \otimes \tilde{\phi}_+({y_2}^1)(y') \\
- 2\, \tilde{\phi}_+(y^1)(y'_1) \otimes y'_2 \otimes y^2 - 2\, y'_1 \otimes \tilde{\phi}_+(y^1)(y'_2) \otimes y^2
\end{aligned}$$

If we apply the mapping p to this last expression, we see that the first and the fourth, the second and the third, the fifth and the eleventh, the sixth and the twelfth, the seventh and the ninth as well as the eighth and the tenth term cancel each other. Here we have used the symmetry condition. We conclude that $p \circ (d^1 g) = 0$, and since p is equivariant, this implies that $p \circ g$ is a 1-cocycle. Since it vanishes on the generators, it vanishes identically by Proposition 5.1.1. This implies that $\tilde{L}_+$ is a Yetter-Drinfel'd Lie algebra. Since the only difference between $\tilde{L}_-$ and $\tilde{L}_+$ lies in the choice of a sign, and the above considerations did not depend of the choice of this sign, $\tilde{L}_-$ is a Yetter-Drinfel'd Lie algebra, too.□

5.5 We now proceed to construct a Lie algebra biform between $\tilde{L}_-$ and $\tilde{L}_+$. We regard the base field $K = \mathbb{C}$ as a trivial $\tilde{L}_+$-module. By Proposition 5.1.1 there is for each $i = 1, \ldots, n$ a unique 1-cocycle ξ_i that satisfies $\xi_i(\tilde{e}_j) = \delta_{ij}\, \epsilon_i$ on the generators $\tilde{e}_j, j = 1, \ldots, n$. ξ_i obviously is an element of the dual vector space $\tilde{L}_+^*$, which is a Lie algebra since $\tilde{L}_+$ is a Lie coalgebra. Since $\tilde{L}_-$ is free, there is a unique Lie algebra homomorphism

$$\xi : \tilde{L}_- \to \tilde{L}_+^*$$

that satisfies $\xi(\tilde{f}_i) = \xi_i$ for $i = 1, \ldots, n$. We now can define the bilinear form

$$\langle \cdot, \cdot \rangle : \tilde{L}_- \times \tilde{L}_+ \to \mathbb{C}, \ (x, y) \mapsto \xi(x)(y)$$

Proposition $\langle \cdot, \cdot \rangle$ is a strict Lie algebra biform.

Proof. We regard $\tilde{L}_+^*$ as an $\tilde{L}_-$-module via

$$\tilde{L}_- \times \tilde{L}_+^* \to \tilde{L}_+^*, (x, y^*) \mapsto x.y^* := [\xi(x), y^*]$$

With respect to this module structure, the mapping $\xi \circ \tilde{\phi}_-(h) : \tilde{L}_- \to \tilde{L}_+^*$ is a 1-cocycle for every $h \in H$ because we have:

$$\begin{aligned} d^1(\xi \circ \tilde{\phi}_-(h))(x \wedge x') &= \xi(\tilde{\phi}_-(h)([x, x'])) - x.\xi(\tilde{\phi}_-(h)(x')) + x'.\xi(\tilde{\phi}_-(h)(x)) \\ &= \xi([\tilde{\phi}_-(h)(x), x'] + [x, \tilde{\phi}_-(h)(x')]) \\ &\quad - [\xi(x), \xi(\tilde{\phi}_-(h)(x'))] + [\xi(x'), \xi(\tilde{\phi}_-(h)(x))] = 0 \end{aligned}$$

Similarly, the mapping $\tilde{\phi}_+^*(h) \circ \xi : \tilde{L}_- \to \tilde{L}_+^*$, where $\tilde{\phi}_+^*(h) : \tilde{L}_+^* \to \tilde{L}_+^*$ denotes the transpose of $\tilde{\phi}_+(h)$, is a 1-cocycle:

$$\begin{aligned} & d^1(\tilde{\phi}_+^*(h) \circ \xi)(x \wedge x')(y) \\ &= \tilde{\phi}_+^*(h)(\xi([x, x']))(y) - [\xi(x), \tilde{\phi}_+^*(h)(\xi(x'))](y) + [\xi(x'), \tilde{\phi}_+^*(h)(\xi(x))](y) \\ &= \xi([x, x'])(\tilde{\phi}_+(h)(y)) - \xi(x)(y_1)\xi(x')(\tilde{\phi}_+(h)(y_2)) + \xi(x')(y_1)\xi(x)(\tilde{\phi}_+(h)(y_2)) \\ &= 0 \end{aligned}$$

In order to show $\xi \circ \tilde{\phi}_-(h) = -\tilde{\phi}_+^*(h) \circ \xi$, it is therefore sufficient to check this equality on the generators $\tilde{f}_1, \ldots, \tilde{f}_n$. But a short calculation shows that both $\xi(\tilde{\phi}_-(h)(\tilde{f}_j))$ and $-\tilde{\phi}_+^*(h)(\xi(\tilde{f}_j))$ are 1-cocycles with values in the base field, and therefore once again it is sufficient to prove their equality on generators, where is holds because of our choice of signs:

$$\begin{aligned} -\tilde{\phi}_+^*(h)(\xi(\tilde{f}_j))(\tilde{e}_i) &= -\xi_j(\tilde{\phi}_+(h)(\tilde{e}_i)) = -\alpha_i(h)\xi_j(\tilde{e}_i) \\ &= -\alpha_j(h)\xi_j(\tilde{e}_i) = \xi(\tilde{\phi}_-(h)(\tilde{f}_j))(\tilde{e}_i) \end{aligned}$$

This implies condition 2.8.1 since we have:

$$\langle \tilde{\phi}_-(h)(x), y \rangle = \xi(\tilde{\phi}_-(h)(x))(y) = -\tilde{\phi}_+^*(h)(\xi(x))(y) = -\langle x, \tilde{\phi}_+(h)(y) \rangle$$

Condition 2.8.4 follows from condition 2.8.1 by a simple calculation. Condition 2.8.3 holds since ξ is a Lie algebra homomorphism:

$$\langle x, y_1 \rangle \langle x', y_2 \rangle = \xi(x)(y_1)\xi(x')(y_2) = [\xi(x), \xi(x')](y) = \xi([x, x'])(y) = \langle [x, x'], y \rangle$$

It remains to show condition 2.8.2. First, note that for any two modules V and W over any Lie algebra the canonical map $V^* \otimes W^* \to (V \otimes W)^*$ is equivariant. Therefore there is no danger of confusion when we consider the two mappings

$$g : \tilde{L}_- \to (\tilde{L}_+ \otimes \tilde{L}_+)^*, x \mapsto (\xi \otimes \xi)(\tilde{\Delta}_-(x))$$

and

$$g' : \tilde{L}_- \to (\tilde{L}_+ \otimes \tilde{L}_+)^*, x \mapsto (y \otimes y' \mapsto \xi(x)([y, y']))$$

Here we consider $\tilde{L}_+$ as a submodule of $\tilde{L}_+^{**}$. The action that arises this way is precisely the coadjoint action: $x.y = \psi_+(x)(y) = y_1\langle x, y_2\rangle$. We first show that $d^1 g = d^1 g'$. We have:

$$\begin{aligned}
(d^1 g)(x \wedge x')(y \otimes y') &= (\xi \otimes \xi)(\tilde{\Delta}_-([x, x']))(y \otimes y') - x.(\xi \otimes \xi)(\tilde{\Delta}_-(x'))(y \otimes y') \\
&\qquad + x'.(\xi \otimes \xi)(\tilde{\Delta}_+(x))(y \otimes y') \\
&= (\xi \otimes \xi)(\tilde{\Delta}_-([x, x']) - x.\tilde{\Delta}_-(x') + x'.\tilde{\Delta}_-(x))(y \otimes y') \\
&= \langle 2\, x^2 \otimes \tilde{\phi}_-(x^1)(x') - 2\, \tilde{\phi}_-(x^1)(x') \otimes x^2, y \otimes y'\rangle \\
&= -2\, \langle x^2, y\rangle\langle x', \tilde{\phi}_+(x^1)(y')\rangle + 2\, \langle x', \tilde{\phi}_+(x^1)(y)\rangle\langle x^2, y'\rangle \\
&= 2\, \langle x, y^2\rangle\langle x', \tilde{\phi}_+(y^1)(y')\rangle - 2\, \langle x', \tilde{\phi}_+(y'^1)(y)\rangle\langle x, y'^2\rangle \\
&= \langle x \otimes x', 2\, y^2 \otimes \tilde{\phi}_+(y^1)(y') - 2\, y'^2 \otimes \tilde{\phi}_+(y'^1)(y)\rangle \\
&= \langle x \otimes x', 2\, y^2 \otimes \tilde{\phi}_+(y^1)(y') - 2\, \tilde{\phi}_+(y^1)(y') \otimes y^2\rangle
\end{aligned}$$

Similarly, we calculate $d^1 g'$:

$$\begin{aligned}
&((d^1 g'))(x \wedge x')(y \otimes y') \\
&\quad = g'([x, x'])(y \otimes y') - (x.g'(x'))(y \otimes y') + (x'.g'(x))(y \otimes y') \\
&\quad = \langle [x, x'], [y, y']\rangle - \langle x, y_1\rangle g'(x')(y_2 \otimes y') - \langle x, y'_1\rangle g'(x')(y \otimes y'_2) \\
&\qquad\quad + \langle x', y_1\rangle g'(x)(y_2 \otimes y') + \langle x', y'_1\rangle g'(x)(y \otimes y'_2) \\
&\quad = \langle x \otimes x', \tilde{\Delta}_+([y, y'])\rangle - \langle x, y_1\rangle\langle x', [y_2, y']\rangle - \langle x, y'_1\rangle\langle x', [y, y'_2]\rangle \\
&\qquad\quad + \langle x', y_1\rangle\langle x, [y_2, y']\rangle + \langle x', y'_1\rangle\langle x, [y, y'_2]\rangle \\
&\quad = \langle x \otimes x', \tilde{\Delta}_+([y, y']) - y.\tilde{\Delta}_+(y') + y'.\tilde{\Delta}_+(y)\rangle \\
&\quad = \langle x \otimes x', 2\, y^2 \otimes \tilde{\phi}_+(y^1)(y') - 2\, \tilde{\phi}_+(y^1)(y') \otimes y^2\rangle
\end{aligned}$$

Since it is easy to see that we have $g(\tilde{f}_j) = 0 = g'(\tilde{f}_j)$, we see that the 1-cocycle $g - g'$ vanishes on generators, and therefore vanishes identically. We therefore have:

$$\langle x_1, y\rangle\langle x_2, y'\rangle = g(x)(y \otimes y') = g'(x)(y \otimes y') = \langle x, [y, y']\rangle$$

which proves the remaining condition. □

5.6 We can now apply the second construction by using Corollary 4.9.1 to obtain a Lie bialgebra $\tilde{L} := \tilde{L}_- \oplus H \oplus \tilde{L}_+$. This is, however, not the Lie algebra we are finally interested in. Nevertheless, we have to gather some information about this Lie algebra before we proceed.

Let us introduce some notation. We denote:

$$e_i = (0, 0, \tilde{e}_i) \qquad \tilde{h} = (0, h, 0) \qquad f_j = (\tilde{f}_j, 0, 0)$$

Proposition 1 We have for $h, h' \in H$:

$$[e_i, f_j] = \delta_{ij}\widetilde{\alpha_i^\vee} \qquad [\tilde{h}, \tilde{h}'] = 0$$
$$[\tilde{h}, e_i] = \alpha_i(h)e_i \qquad [\tilde{h}, f_j] = -\alpha_j(h)f_j$$

and, together with the relations

$$\widetilde{h + h'} = \tilde{h} + \tilde{h}' \qquad \widetilde{\lambda h} = \lambda\tilde{h}$$

for $\lambda \in \mathbb{C}$, this is a presentation of $\tilde{L}$ in terms of generators and relations.

Proof. It is easy to verify the above relations. We therefore only prove the first one, which is also the most difficult one. We have:

$$\begin{aligned}[e_i, f_j] = [(0, 0, \tilde{e}_i), (\tilde{f}_j, 0, 0)] &= (\psi_-(\tilde{e}_i)(\tilde{f}_j), \{\tilde{f}_j, \tilde{e}_i\}, -\psi_+(\tilde{f}_j)(\tilde{e}_i)) \\ &= (0, \frac{1}{\epsilon_i}\alpha_i^\vee\langle\tilde{f}_j, \tilde{e}_i\rangle, 0) = \delta_{ij}(0, \alpha_i^\vee, 0)\end{aligned}$$

To see that this really is a presentation in terms of generators and relations, denote by $\tilde{L}'$ the Lie algebra with the above presentation. We then have a surjective map $f : \tilde{L}' \to \tilde{L}$. We then can apply [9], Theorem 1.2, p. 3 to obtain a triangular decomposition of $\tilde{L}'$ into H and two free Lie algebras in such a way that f preserves the triangular decomposition and restricts to an isomorphism in every summand. Therefore, it is also itself an isomorphism. □

We now introduce an invariant bilinear form on $\tilde{L}$:

$$\langle\cdot,\cdot\rangle_K : \tilde{L} \times \tilde{L} \to \mathbb{C}, ((x, h, y), (x', h', y')) \mapsto \langle x, y'\rangle + \langle h, h'\rangle_H + \langle x', y\rangle$$

Proposition 2 $\langle\cdot,\cdot\rangle_K$ is the unique symmetric, invariant bilinear form on $\tilde{L}$ that has the following properties:

1. $\tilde{L}_+$ and $\tilde{L}_-$ are totally isotropic, i. e. the restriction of the bilinear form to these subspaces is zero.
2. H is orthogonal to $\tilde{L}_+$ and to $\tilde{L}_-$.
3. For all $h, h' \in H$ we have: $\langle\tilde{h}, \tilde{h}'\rangle_K = \langle h, h'\rangle_H$
4. For all $i, j = 1, \ldots, n$ we have: $\langle e_i, f_j\rangle_K = \delta_{ij}\epsilon_i$

Proof. We prove first that the bilinear form is invariant. We have:

$$\begin{aligned}
&\langle [(x,h,y),(x',h',y')],(x'',h'',y'')\rangle_K = \\
&\langle [x,x'],y''\rangle + \langle \tilde{\phi}_-(h)(x') - \tilde{\phi}_-(h')(x),y''\rangle + \langle \psi_-(y)(x') - \psi_-(y')(x),y''\rangle \\
&\ + \langle \{x',y\} - \{x,y'\},h''\rangle_H + \langle x'',[y,y']\rangle + \langle x'',\tilde{\phi}_+(h)(y') - \tilde{\phi}_+(h')(y)\rangle \\
&\ + \langle x'',\psi_+(x)(y') - \psi_+(x')(y)\rangle
\end{aligned}$$

On the other hand, we have:

$$\begin{aligned}
&\langle (x,h,y),[(x',h',y'),(x'',h'',y'')]\rangle_K = \\
&\langle x,[y',y'']\rangle + \langle x,\tilde{\phi}_+(h')(y'') - \tilde{\phi}_+(h'')(y')\rangle + \langle x,\psi_+(x')(y'') - \psi_+(x'')(y')\rangle \\
&\ + \langle h,\{x'',y'\} - \{x',y''\}\rangle_H + \langle [x',x''],y\rangle + \langle \tilde{\phi}_-(h')(x'') - \tilde{\phi}_-(h'')(x'),y\rangle \\
&\ + \langle \psi_-(y')(x'') - \psi_-(y'')(x'),y\rangle
\end{aligned}$$

In these expressions, the terms which only contain brackets and the actions ψ_- and ψ_+ are equal, since by the elementary properties of the coadjoint actions we can bring them to the other side as a bracket. The terms involving h' are also equal in both expressions by the properties of Lie algebra biforms. We are therefore left with proving the equality:

$$\begin{aligned}
&\langle \tilde{\phi}_-(h)(x'),y''\rangle + \langle \{x',y\} - \{x,y'\},h''\rangle_H + \langle x'',\tilde{\phi}_+(h)(y')\rangle \\
&\qquad = -\langle x,\tilde{\phi}_+(h'')(y')\rangle + \langle h,\{x'',y'\} - \{x',y''\}\rangle_H - \langle \tilde{\phi}_-(h'')(x'),y\rangle
\end{aligned}$$

But this equality follows from:

$$\langle x,\tilde{\phi}_+(h)(y)\rangle = 2\,\langle h,y^1\rangle_H \langle x,y^2\rangle = \langle h,\{x,y\}\rangle_H$$

It is obvious that $\langle \cdot,\cdot\rangle_K$ is symmetric and satisfies the remaining of the stated conditions. Let us now assume that there is another symmetric invariant bilinear form $\langle \cdot,\cdot\rangle_I$ that also satisfies these properties. We consider $\tilde{L}$ and $\tilde{L}^*$ as $\tilde{L}$-modules via the adjoint resp. coadjoint action. Define:

$$\eta_I : \tilde{L} \to \tilde{L}^*, z \mapsto (z' \mapsto \langle z,z'\rangle_I)$$

The invariance of the bilinear form then can be reformulated by saying that η_I is equivariant. Since equivariant maps are determined by their values on a system of generators of the module, it is sufficient to prove $\langle z,z'\rangle_I = \langle z,z'\rangle_K$ for $z = e_i, f_j$ or $\tilde{h}$. We consider first the case $z = e_i$. Then both sides of the above equation are zero if $z' \in \tilde{L}_+$ or $z' \in H$. We therefore can assume that $z' \in \tilde{L}_-$. Consider the mapping

$$\zeta_i : \tilde{L}_- \to \mathbb{C}, x \mapsto \langle (0,0,\tilde{e}_i),(x,0,0)\rangle_I$$

We prove that this mapping is a 1-cocycle, which simply means that it vanishes on brackets:

$$\begin{aligned}
\zeta_i([x,x']) &= \langle (0,0,\tilde{e}_i),[(x,0,0),(x',0,0)]\rangle_I \\
&= \langle [(0,0,\tilde{e}_i),(x,0,0)],(x',0,0)\rangle_I \\
&= \langle (0,0,-\psi_+(x)(\tilde{e}_i)),(x',0,0)\rangle_I \\
&= 0
\end{aligned}$$

since by our construction of the coproduct we have $\psi_+(x)(\tilde{e}_i) = 0$. Since we know by Proposition 5.1.1 that cocycles are determined by their values on generators, it is sufficient to prove $\langle e_i, f_j \rangle_I = \langle e_i, f_j \rangle_K$, which is one of our assumptions.

The case $z = f_j$ can be treated similarly, and the case $z = \tilde{h}$ is easy: Since by assumption both sides of the equation vanish if $z' \in \tilde{L}_+$ or $z' \in \tilde{L}_-$, we can assume $z' = \tilde{h}'$, and again by assumption we have that:

$$\langle \tilde{h}, \tilde{h}' \rangle_I = \langle h, h' \rangle_H = \langle \tilde{h}, \tilde{h}' \rangle_K$$

□

5.7 We are now in a position where we can construct symmetrizable Kac-Moody algebras on the basis of the second construction. Define:

$$R_- := \{x \in \tilde{L}_- | \forall y \in \tilde{L}_+ : \langle x, y \rangle = 0\} \qquad R_+ := \{y \in \tilde{L}_+ | \forall x \in \tilde{L}_- : \langle x, y \rangle = 0\}$$

and set:

$$L_+ := \tilde{L}_+ / R_+ \qquad L_- := \tilde{L}_- / R_-$$

As explained in Subsection 2.10, there is a canonical nondegenerate Lie biform between L_- and L_+, and therefore we can use the second construction to obtain a Lie bialgebra

$$L := L_- \oplus H \oplus L_+$$

We claim that this Lie algebra is isomorphic to the Kac-Moody algebra corresponding to the symmetrizable matrix A as defined in [9], which we denote by L'. This can be seen as follows: Denote by f the mapping

$$f : \tilde{L} \to L, (x, h, y) \mapsto (\bar{x}, h, \bar{y})$$

The Lie algebra L also carries an invariant symmetric bilinear form:

$$\langle \cdot, \cdot \rangle_{\bar{K}} : L \times L \to \mathbb{C}, ((\bar{x}, h, \bar{y}), (\bar{x}', h', \bar{y}')) \mapsto \langle x, y' \rangle + \langle h, h' \rangle_H + \langle x', y \rangle$$

which is nondegenerate by construction of L and is preserved by f. The Kac-Moody algebra L' also carries an invariant symmetric nondegenerate bilinear form $(\cdot|\cdot)$ by [9], Theorem 2.2, p. 17. It follows from Proposition 5.6.1 that L' arises as a quotient of $\tilde{L}$. Denote by $p : \tilde{L} \to L'$ the corresponding projection. We then can introduce a second bilinear form $\langle \cdot, \cdot \rangle_I$ on $\tilde{L}$ by pullback via p:

$$\langle z, z' \rangle_I := (p(z)|p(z'))$$

It follows from [9], Theorem 2.2, p. 17 and equation (2.2.1), p. 18 that $\langle \cdot, \cdot \rangle_I$ has the properties stated in Proposition 5.6.2, and therefore is equal to $\langle \cdot, \cdot \rangle_K$. If we define:

$$R := \{z \in \tilde{L} | \forall z' \in \tilde{L} : \langle z, z' \rangle_K = 0\},$$

then the maps f and p induce isomorphisms

$$\bar{f} : \tilde{L}/R \to L \qquad \bar{p} : \tilde{L}/R \to L'$$

Therefore, $\bar{p} \circ (\bar{f})^{-1}$ is a Lie algebra isomorphism between L and L'. □

5.8 We assume now that the matrix A we are using is a Cartan matrix. It will then happen that L_+ and L_- are finite dimensional. In this case, the resulting Lie bialgebra is quasitriangular, for the following reason: We have already explained in Subsection 2.5 that coquasitriangular Lie bialgebras are quasitriangular (and vice versa) if they are finite dimensional and their dual r-matrix is nondegenerate. We also have explained in Subsection 2.12 how the two variants of the second construction are equivalent in this case: If we set $k = \dim H = 2n - \operatorname{rank}(A)$ and choose dual bases $a_1, \ldots, a_k$ and $b_1, \ldots, b_k$ of H with respect to the dual r-matrix $\langle\cdot,\cdot\rangle_0$, that is, bases satisfying $\langle b_i, a_j\rangle_0 = \delta_{ij}$, then H is also quasitriangular with r-matrix $r = \sum_{i=1}^k a_i \otimes b_i$, and inducing actions $\tilde{\phi}_+$ and $\tilde{\phi}_-$ by a coactions $\tilde{\delta}_+$ and $\tilde{\delta}_-$ via the formulas

$$\tilde{\phi}_+(h)(y) = \langle y^1, h\rangle_0 y^2 \qquad \tilde{\phi}_-(h)(x) = \langle h, x^1\rangle_0 x^2$$

is the same as inducing coactions by actions via the formulas:

$$\tilde{\delta}_+(y) = \sum_{i=1}^k b_i \otimes \tilde{\phi}_+(a_i)(y) \qquad \tilde{\delta}_-(x) = \sum_{i=1}^k a_i \otimes \tilde{\phi}_-(b_i)(x)$$

Therefore, it follows from Corollary 4.8 that L is quasitriangular with classical r-matrix

$$r' := \sum_{i=1}^k (0, a_i, 0) \otimes (0, b_i, 0) + \sum_{j=1}^m (\bar{x}_j, 0, 0) \otimes (0, 0, \bar{y}_j)$$

where we have chosen bases $\bar{x}_1, \ldots, \bar{x}_m \in L_-$ and $\bar{y}_1, \ldots, \bar{y}_m \in L_+$ that satisfy $\langle x_i, y_j\rangle = \delta_{ij}$. Moreover, it follows that symmetrizing the classical r-matrix yields the Casimir operator. We remark that in the infinite dimensional case L still has a generalized r-matrix - it is called 'pseudotriangular' (cf. [5], p. 805).

Acknowledgement This article was presented at the Ring theory conference, Miskolc, Hungary, July 21-26, 1996. The author thanks the 'Graduiertenkolleg: Mathematik im Bereich ihrer Wechselwirkung mit der Physik' of the Deutsche Forschungsgemeinschaft for financial support making the talk possible.

References

[1] H. Cartan / S. Eilenberg: Homological algebra, Princeton Math. Ser. 19, Princeton University Press, Princeton 1956

[2] V. Chari / A. Pressley: A guide to quantum groups, Cambridge University Press, Cambridge 1994

[3] C. Chevalley / S. Eilenberg: Cohomology theory of Lie groups and Lie algebras, Trans. Amer. Math. Soc. 63 (1948), 85-124

[4] V. G. Drinfel'd: Hamiltonian structures on Lie groups, Lie bialgebras and the geometric meaning of the classical Yang-Baxter equation, Dokl. Akad.

Nauk SSSR 268 (1983), 285-287. English translation: Soviet Math. Dokl. 27 (1983), 68-70

[5] V. G. Drinfel'd: Quantum groups; In: Proceedings of the International Congress of Mathematicians, Volume I, 798-820, AMS, Berkeley 1987

[6] O. Gabber / V. G. Kac: On defining relations of certain infinite-dimensional Lie algebras, Bull. Amer. Math. Soc. (N.S.) 5 (1981), 185-189

[7] P. J. Hilton / U. Stammbach: A course in homological algebra, Grad. Texts in Math. 4, Springer Verlag, Berlin 1971

[8] V. G. Kac: Simple graded Lie algebras of finite growth, Funktsional. Anal. i. Prilozhen. 1 (1967), No. 4, 83-83. English translation: Funct. Anal. Appl. 1 (1967), 328-329

[9] V. G. Kac: Infinite dimensional Lie algebras, 3. ed., Cambridge University Press, Cambridge 1990

[10] R. V. Moody: Lie algebras associated with generalized Cartan matrices, Bull. Amer. Math. Soc. 73 (1967), 217-221

[11] R. V. Moody / A. Pianzola: Lie algebras with triangular decompositions, Canad. Math. Soc. Ser. Monogr. Advanced Texts, Wiley 1995

[12] B. Pareigis: On Lie algebras in braided categories, Preprint gk-mp-9506/20, Munich 1995, to appear in: Banach Center Publ., Vol. 40 (1997)

[13] B. Pareigis: On Lie algebras in the category of Yetter-Drinfel'd modules, to appear in: Appl. Categ. Structures (1997)

[14] C. Reutenauer: Free Lie algebras, London Math. Soc. Monogr. (N.S.) 7, Oxford University Press, Oxford 1993

[15] Y. Sommerhäuser: Deformierte universelle Einhüllende, Diplomarbeit, Munich 1994

[16] Y. Sommerhäuser: Deformed enveloping algebras, New York J. Math. 2 (1996), 35-58

[17] M. Sweedler: Hopf algebras, W. A. Benjamin, New York 1969

[18] M. Takeuchi: Matched pairs of groups and bismash products of Hopf algebras, Comm. Algebra 9 (1981), 841-882

[19] C. A. Weibel: An introduction to homological algebra, Cambridge Stud. Adv. Math. 38, Cambridge University Press, Cambridge 1994