Chow motives of twisted flag varieties

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Abstract

Let $G$ be an adjoint simple algebraic group of inner type. We express the Chow motive (with integral coefficients) of an anisotropic projective $G$-homogeneous variety in terms of motives of simpler $G$-homogeneous varieties, namely, those that correspond to maximal parabolic subgroups of $G$. We decompose the motive of a generalized Severi-Brauer variety $SB_2(A)$ of a division algebra $A$ of degree 5 into a direct sum of twisted motives of the Severi-Brauer variety $SB(B)$ of a division algebra $B$ Brauer-equivalent to the tensor square $A^{\otimes 2}$. As an application we provide another counter-example to the uniqueness of a direct sum decomposition in the category of motives with integral coefficients.

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1 Introduction

Let $G$ be an adjoint simple algebraic group of inner type over a field $F$. Let $X$ be a twisted flag variety, i.e., a projective $G$-homogeneous variety over $F$. The main purpose of the paper is to express the Chow motive of $X$ in terms of motives of “minimal” flags, i.e., those $G$-homogeneous varieties that correspond to maximal parabolic subgroups of $G$.

Observe that the motive of an isotropic $G$-homogeneous variety can be decomposed in terms of motives of simpler $G$-homogeneous varieties using the techniques developed by Chernousov, Gille, Merkurjev [CGM05] and Karpenko [Ka01]. For $G$-varieties, when $G$ is isotropic, one obtains a similar decomposition following the arguments of Brosnan [Br03]. In the case

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of $G$-varieties, where $G$ is anisotropic, no general decomposition methods are known except several particular cases of quadrics (see for example Rost \cite{Ro98}) Severi-Brauer varieties (see Karpenko \cite{Ka95}) and exceptional varieties of type $F_4$ (see \cite{NSZ05}).

In the present paper we provide methods that allow to decompose the motives of some anisotropic twisted flag $G$-varieties, where the root system of $G$ is of types $A_n$, $B_n$, $C_n$, $G_2$ and $F_4$, i.e., has a Dynkin diagram which does not branch.

As an application, we provide another counter-example to the uniqueness of a direct sum decomposition in the category of Chow motives with integral coefficients (see \ref{CM04}). Observe that such a counter-example was already given in \cite{CM04} Example 9.4 using a $G$-homogeneous variety, where $G$ is a product of two simple groups. Our example is given by a $G$-variety, where $G$ is a simple group. Apart from this, our example involves a motivic decomposition

$$\mathcal{M}(\text{SB}_2(A)) \simeq \mathcal{M}(\text{SB}(B)) \oplus \mathcal{M}(\text{SB}(B))(2),$$

of the motive of a generalized Severi-Brauer variety $\text{SB}_2(A)$ of a division algebra $A$ of degree 5 into a direct sum of twisted motives of the Severi-Brauer variety $\text{SB}(B)$ of a division algebra $B$ Brauer-equivalent to the tensor square $A \otimes 2$. Observe that the motive $\mathcal{M}(\text{SB}(B))$ is isomorphic to the motive $\mathcal{M}(\text{SB}(A))$ over $\mathbb{Z}[\frac{1}{2}]$ and $\mathbb{Z}[\frac{1}{3}]$, but not integrally.

The paper is organized as follows. In section \ref{MainResults} we state the main results. We then provide some technical facts that are extensively used in the proofs (section \ref{TechnicalFacts}). In the other sections we give proofs of the results for varieties of type $A_n$ (section \ref{An}), of types $B_n$ and $C_n$ (section \ref{BnCn}), and exceptional varieties of types $G_2$ and $F_4$ (section \ref{G2F4}). Section \ref{MotivicDecomposition} is devoted to the motivic decomposition of generalized Severi-Brauer varieties.

**Notation and Conventions** By $G$ we denote an adjoint simple algebraic group of inner type over a field $F$ and by $n$ its rank. By $F_s$ we denote the separable closure of $F$ and by $X_s$ the respective base change $X_s = X \times_F F_s$ of a variety $X$. All varieties that appear in the paper are projective $G$-homogeneous varieties over $F$. They are twisted forms of the varieties $G'/P$, where $G'$ is the split adjoint simple group of the same type as $G$ and $P$ its parabolic subgroup. The Chow motive of a variety $X$ is denoted by $\mathcal{M}(X)$. By $A$ we denote a central simple algebra over $F$ of index $\text{ind}(A)$ and by $\text{SB}(A)$ the corresponding Severi-Brauer variety. $I$ is always a right ideal of $A$ and $\text{rdim} I$ stands for its reduced dimension. $V$ is a vector space over $F$. 

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By $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ we denote a partition $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_l \geq 0$ with $|\lambda| = \lambda_1 + \lambda_2 + \ldots + \lambda_l$. Integers $d_1, d_2, \ldots, d_k$ always satisfy the condition $1 \leq d_1 < d_2 < \ldots < d_k \leq n$ and are the dimensions of some flag. For each $i = 0, \ldots, k$ we define $\delta_i$ to be the difference $d_{i+1} - d_i$ (assuming here $d_0 = 0$ and $d_{k+1} = n + 1$).

2 Statements of Results

We follow [MPW96, Appendix] and [CG05] for the description of projective $G$-homogeneous varieties that appear below. According to the type of the group $G$, we obtain the following results.

$A_n$: In this case $G = \text{PGL}_1(A)$, where $A$ is a central simple algebra of degree $n + 1$, $n > 0$, and the set of $F$-points of a projective $G$-homogeneous variety $X$ can be identified with the set of flags of (right) ideals

$$X(d_1, \ldots, d_k) = \{I_1 \subset I_2 \subset \ldots \subset I_k \subset A\}$$

of fixed reduced dimensions $1 \leq d_1 < d_2 < \ldots < d_k \leq n$. Observe that this variety is a twisted form of $G'/P$, where $G' = \text{PGL}_{n+1}$ and $P$ is the standard parabolic subgroup corresponding to the simple roots on the Dynkin diagram, numbered by $d_i$.

\[
\begin{array}{cccccc}
1 & 2 & 3 & n-2 & n-1 & n \\
\end{array}
\]

The following result reduces the computation of the motive of $X$ to the motives of “smaller” flags

2.1 Theorem. Suppose that $\gcd(\text{ind}(A), d_1, \ldots, \hat{d}_m, \ldots, d_k) = 1$, then

$$\mathcal{M}(X(d_1, \ldots, d_k)) \simeq \bigoplus_{\lambda} \mathcal{M}(X(d_1, \ldots, \hat{d}_m, \ldots, d_k))(\delta_m \delta_{m-1} - |\lambda|),$$

where the sum is taken over all partitions $\lambda = (\lambda_1, \ldots, \lambda_{\delta_{m-1}})$ such that $\delta_m \geq \lambda_1 \geq \ldots \geq \lambda_{\delta_{m-1}} \geq 0$.

Proof. See 4.8.

As a consequence, for the variety of complete flags we obtain
2.2 Corollary. The motive of the variety $X = X(1, \ldots, n)$ of complete flags is isomorphic to

$$\mathcal{M}(X) \simeq \bigoplus_{i=0}^{n(n-1)/2} \mathcal{M}(\text{SB}(A))(i)^{\oplus a_i},$$

where $a_i$ are the coefficients of the polynomial $\varphi_n(z) = \sum_i a_i z^i = \prod_{k=2}^n \frac{z^k - 1}{z - 1}$.

Proof. Apply Theorem 2.1 recursively to the sequence of varieties $X(1, \ldots, n)$, $X(1, \ldots, n-1)$, ..., $X(1, 2)$ and $X(1) = \text{SB}(A)$.

Another interesting example is the “incidence” variety $X(1, n)$:

2.3 Corollary. The motive of $X(1, n)$ is isomorphic to

$$\mathcal{M}(X(1, n)) \simeq \bigoplus_{i=0}^{n-1} \mathcal{M}(\text{SB}(A))(i).$$

In order to complete the picture we need to know how to decompose the motive of a “minimal” flag, i.e., a generalized Severi-Brauer variety.

Note that for some rings of coefficients (fields, discrete valuation rings) one easily obtains the desired decomposition by using Krull-Schmidt Theorem (the uniqueness of a direct sum decomposition). More precisely, consider the subcategory $\mathcal{M}(G, R)$ of the category of motives with coefficients in a ring $R$ that is a pseudo-abelian completion of the category of motives of projective $G$-homogeneous varieties (see [CM04, section 8]). Then we have the following

2.4 Proposition. Let $X(d) = \text{SB}_d(A)$, $1 < d < n$, be a generalized Severi-Brauer variety for a central simple algebra $A$ of degree $n + 1$ such that $\gcd(\text{ind}(A), d) = 1$. Let $R$ be a ring such that Krull-Schmidt Theorem holds in the category $\mathcal{M}(G, R)$. Then the motive of $\text{SB}_d(A)$ with coefficients in $R$ is isomorphic to

$$\mathcal{M}(\text{SB}_d(A)) \simeq \bigoplus_{i \in \mathcal{I}} \mathcal{M}(\text{SB}(A))(i)^{\oplus a_i},$$

where the integers $a_i$ are the coefficients of the polynomial $\frac{\varphi_n(z)}{\varphi_d(z)\varphi_{n+1-d}(z)}$ at terms $z^i$ and the set of indices $\mathcal{I} = \{i \mid a_i \neq 0\}$.

Proof. See 4.10. □
It turns out that the motives of some generalized Severi-Brauer varieties with integral coefficients can still be decomposed.

**2.5 Theorem.** Let \( \text{SB}_2(A) \) be a generalized Severi-Brauer variety for a division algebra \( A \) of degree 5. Then there is an isomorphism

\[
\mathcal{M}(\text{SB}_2(A)) \simeq \mathcal{M}(\text{SB}(B)) \oplus \mathcal{M}(\text{SB}(B))(2),
\]

where \( B \) is a division algebra Brauer-equivalent to the tensor square \( A^\otimes 2 \).

*Proof.* See [5.11].

**2.6 Remark.** It is expected that the mod-\( p \) version of this Theorem can also be proven using techniques dealing with norm varieties. Namely, if the algebra \( A \) is cyclic, then it corresponds to a symbol in \( K_2^M(F)/5 \) which is split by the variety \( \text{SB}_2(A) \) (see [Su05]). By the results of Voevodsky [Vo03] the motive of \( \text{SB}_2(A) \) with \( \mathbb{Z}/5\mathbb{Z} \)-coefficients splits.

As an immediate consequence of Theorems 2.1 and 2.5 we obtain

**2.7 Corollary.** The Krull-Schmidt Theorem fails in the category of motives \( \mathcal{M}(\text{PGL}_1(A), \mathbb{Z}) \) where \( A \) is a division algebra of degree 5.

*Proof.* Apply Theorem 2.1 recursively to the sequences of varieties \( X(1, 2), X(1) \) and \( X(1, 2), X(2) \), where \( X(1, 2) \) is the twisted flag \( G \)-variety for \( G = \text{PGL}_1(A) \). We obtain two decompositions of the motive of \( X(1, 2) \)

\[
\bigoplus_{i=0}^{3} \mathcal{M}(\text{SB}(A))(i) \simeq \mathcal{M}(X(1, 2)) \simeq \mathcal{M}(\text{SB}_2(A)) \oplus \mathcal{M}(\text{SB}_2(A))(1).
\]

Apply now Theorem 2.5 to the components of the second decomposition. We obtain two decompositions of the motive \( \mathcal{M}(X(1, 2)) \)

\[
\bigoplus_{i=0}^{3} \mathcal{M}(\text{SB}(A))(i) \simeq \mathcal{M}(X(1, 2)) \simeq \bigoplus_{i=0}^{3} \mathcal{M}(\text{SB}(B))(i). \quad (*)
\]

By [Ka95, Theorem. 2.2.1] and [Ka00, Criterion 7.1] the motives \( \mathcal{M}(\text{SB}(A)) \) and \( \mathcal{M}(\text{SB}(B)) \) are indecomposable and non-isomorphic. This finishes the proof of the corollary.
2.8 Remark. Observe that the counter-example provided by Chernousov and Merkurjev (see [CM04, Example 9.4]) is the product of two Severi-Brauer varieties $X = \text{SB}(A) \times \text{SB}(B)$ which is a $G$-homogeneous variety for the semi-simple group $G = \text{PGL}_1(A) \times \text{PGL}_1(B)$, where $A$ and $B$ are two division algebras of degree 5 generating the same subgroup in the Brauer group. The example that we provide, i.e., the flag $X(1, 2)$, is a $G$-homogeneous variety for the simple group $G = \text{PGL}_1(A)$. Moreover, it implies that the cancellation property fails in the category of Chow motives $\mathcal{M}(G, \mathbb{Z})$. Indeed, from one hand side we have two different decompositions into indecomposable objects according to [CM04, Example 9.4]

$$
\bigoplus_{i=0}^{4} \mathcal{M}(\text{SB}(A))(i) \cong \mathcal{M}(\text{SB}(A) \times \text{SB}(B)) \cong \bigoplus_{i=0}^{4} \mathcal{M}(\text{SB}(B))(i),
$$

where $B$ is a division algebra of degree 5 Brauer-equivalent to $A^\otimes 2$. From another hand side we have two decompositions (*) of Corollary 2.7.

$B_n$: We assume that the characteristic of the base field $F$ is different from 2. It is known that $G = \text{O}^+(V, q)$, where $(V, q)$ is a regular quadratic space of dimension $2n + 1$, $n > 0$, and projective $G$-homogeneous varieties can be described as flags of totally $q$-isotropic subspaces

$$X(d_1, \ldots, d_k) = \{V_1 \subset \ldots \subset V_k \subset V\}.$$

of fixed dimensions $1 \leq d_1 < \ldots < d_k \leq n$. Observe that this variety is a twisted form of $G'/P$, where $G'$ is a split group of the same type as $G$ and $P$ is the standard parabolic subgroup corresponding to the simple roots on the Dynkin diagram, numbered by $d_i$.

The following result shows that some motives of flag varieties can be decomposed into a direct sum of twisted motives of “smaller” flags.

2.9 Theorem. Suppose that $m < k$, then

$$\mathcal{M}(X(d_1, \ldots, d_k)) \cong \bigoplus_{\lambda} \mathcal{M}(X(d_1, \ldots, \hat{d}_m, \ldots, d_k))(\delta_m \delta_{m-1} - |\lambda|),$$

where the sum is taken over all partitions $\lambda = (\lambda_1, \ldots, \lambda_{\delta_{m-1}})$ such that $\delta_m \geq \lambda_1 \geq \ldots \geq \lambda_{\delta_{m-1}} \geq 0$. 

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Proof. See \[6.5\] \hfill \square

In particular, for the variety of complete flags we obtain a formula similar to the one of Corollary \[2.2\].

2.10 Corollary. The motive of the variety of complete flags \(X = X(1, 2, \ldots, n)\) is isomorphic to

\[
\mathcal{M}(X) \simeq \bigoplus_{i=0}^{n(n-1)/2} \mathcal{M}(X(n))(i)^{\oplus a_i},
\]

where the \(a_i\) are the coefficients of the polynomial \(\varphi_n(z) = \sum_i a_i z^i = \prod_{k=2}^n \frac{z^k - 1}{z-1}\). and \(X(n)\) is the twisted form of the maximal orthogonal Grassmannian.

\[C_n:\] We assume that the characteristic of the base field \(F\) is different from 2. In this case \(G = \text{Aut}(A, \sigma)\), where \(A\) is a central simple algebra of degree \(2n, n \geq 2\), with an involution \(\sigma\) of symplectic type on \(A\), and a projective \(G\)-homogeneous variety can be described as the set of flags of (right) ideals

\[
X(d_1, \ldots, d_k) = \{I_1 \subset \cdots \subset I_k \subset A \mid I_i \subseteq I_i^\perp\}
\]

of fixed reduced dimensions \(1 \leq d_1 < \cdots < d_k \leq n\), where \(I^\perp = \{x \in A \mid \sigma(x)I = 0\}\) is the right ideal of reduced dimension \(2n - \text{rdim} I\). Observe that this variety is a twisted form of \(G'/P\), where \(P\) is the standard parabolic subgroup corresponding to the simple roots on the Dynkin diagram, numbered by \(d_i\).

\[
\circ \circ \cdots \circ \circ \circ \circ \circ \circ \circ
\]

1 2 3 \(n-2\) \(n-1\) \(n\)

Again, the motives of some flag varieties can be decomposed into a direct sum of twisted motives of “smaller” flags.

2.11 Theorem. Suppose that \(d_i\) is odd for some \(i < k\) and \(d_k - d_{k-1} = 1\). Then

\[
\mathcal{M}(X(d_1, \ldots, d_k)) \simeq \bigoplus_{i=0}^{2n-2d_k-1} \mathcal{M}(X(d_1, \ldots, d_{k-1}))(i).
\]

In particular, for the variety of complete flags we obtain
2.12 Corollary. The motive of the variety of complete flags $X = X(1, 2, \ldots, n)$ is isomorphic to
\[\mathcal{M}(X(1, \ldots, n)) \simeq \bigoplus_{i=0}^{n(n-1)} \mathcal{M}(\text{SB}(A))(i)^{\oplus a_i},\]
where $a_i$ are the coefficients of the polynomial $\psi_n(z) = \prod_{k=1}^{n-1} \frac{z^{2k-1}}{z-1}$.

$G_2$: We suppose that the characteristic of $F$ is not 2. It is known that $G = \text{Aut}(C)$, where $C$ is a Cayley algebra over $F$. By an $i$-space, where $i = 1, 2$, we mean an $i$-dimensional subspace $V_i$ of $C$ such that $uv = 0$ for every $u, v \in V_i$. The only flag variety corresponding to a non-maximal parabolic subgroup is the variety of complete flags $X(1, 2)$ which is described as follows
\[X(1, 2) = \{V_1 \subset V_2 \mid V_i \text{ is a } i\text{-subspace of } C\}.
\]
We enumerate the simple roots on the Dynkin diagram as follows:
\[
\begin{array}{c}
\circ \\
\circ \\
\circ \\
\circ \\
\end{array}
\]

In this case we obtain

2.13 Theorem. The motive of the variety of complete flags $X = X(1, 2)$ is isomorphic to
\[\mathcal{M}(X) \simeq \mathcal{M}(X(2)) \oplus \mathcal{M}(X(2))(1).\]

Proof. See [7.5]

Observe that by the result of Bonnet [Bo03] the motives of $X(1)$ and $X(2)$ are isomorphic (here $X(1)$ is a 5-dimensional quadric).

$F_4$: We suppose that the characteristic of $F$ is neither 2 nor 3. It is known that $G = \text{Aut}(J)$, where $J$ is an exceptional Jordan algebra of dimension 27 over $F$. Set $\mathcal{I} = \{1, 2, 3, 6\}$. By an $i$-space, $i \in \mathcal{I}$, we mean an $i$-dimensional subspace $V$ of $J$ such that every $u, v \in V$ satisfy the following condition:
\[\text{Tr}(u) = 0, \ u \times v = 0, \text{ and if } i < 6 \text{ then } u(va) = v(ua) \text{ for all } a \in J.\]
A projective $G$-homogeneous variety can be described as the set of flags of subspaces

$$X(d_1, \ldots, d_k) = \{ V_1 \subset \ldots \subset V_k \mid V_i \text{ is a } d_i\text{-subspace of } J \}.$$  

where the integers $d_1 < \ldots < d_k$ are taken from the set $\mathcal{I}$. Observe that this variety is a twisted form of $G'/P$, where $P$ is the standard parabolic subgroup corresponding to the simple roots on the Dynkin diagram, numbered by $d_i$.

In this case we obtain

**2.14 Theorem.** Suppose that $m < k$ and either $d_{m+1} < 6$ or $d_m = 1$, then

$$\mathcal{M}(X(d_1, \ldots, d_k)) \simeq \bigoplus_{\lambda} \mathcal{M}(X(d_1, \ldots, \hat{d}_m, \ldots, d_k)) (\delta_m \delta_{m-1} - |\lambda|),$$

where the sum is taken over all partitions $\lambda = (\lambda_1, \ldots, \lambda_{\delta_m-1})$ such that $\delta_m \geq \lambda_1 \geq \ldots \geq \lambda_{\delta_m-1} \geq 0$.

**Proof.** See [7.10].

### 3 Preliminaries

In the present section, we introduce the category of Chow motives following [Ma68]. We formulate the Grassmann Bundle Theorem (see [Ko91, Theorem 3.2]) and recall the notion of a functor of points following [Ka01, section 8].

**3.1 (Chow motives).** Let $F$ be a field and $\text{Var}_F$ be the category of smooth projective varieties over $F$. We define the category $\text{Cor}_F$ of correspondences over $F$. Its objects are non-singular projective varieties over $F$. For morphisms, called correspondences, we set $\text{Mor}(X, Y) := \text{CH}^{\dim X}(X \times Y)$. For any two correspondences $\alpha \in \text{CH}(X \times Y)$ and $\beta \in \text{CH}(Y \times Z)$ we define the composition $\beta \circ \alpha \in \text{CH}(X \times Z)$

$$\beta \circ \alpha = \text{pr}_{13*}(\text{pr}_{12}^*(\alpha) \cdot \text{pr}_{23}^*(\beta)),$$
where $\text{pr}_{ij}$ denotes the projection on product of the $i$-th and $j$-th factors of $X \times Y \times Z$ respectively and $\text{pr}_{ij}^*$, $\text{pr}_{ij}^*$ denote the induced push-forwards and pull-backs for Chow groups. Observe that the composition $\circ$ induces a ring structure on the abelian group $\text{CH}^{\dim X}(X \times X)$. The unit element of this ring is the class of the diagonal $\Delta_X$.

The pseudo-abelian completion of $\text{Cor}_F$ is called the category of Chow motives and is denoted by $\mathcal{M}_F$. The objects of $\mathcal{M}_F$ are pairs $(X, p)$, where $X$ is a non-singular projective variety and $p$ is a projector, that is, $p \circ p = p$. The motive $(X, \Delta_X)$ will be denoted by $\mathcal{M}(X)$.

3.2. By the construction $\mathcal{M}_F$ is a self-dual tensor additive category, where the duality is given by the transposition of cycles $\alpha \mapsto \alpha^t$ and the tensor product is given by the usual fiber product $(X, p) \otimes (Y, q) = (X \times Y, p \times q)$. Moreover, the Chow functor $\text{CH} : \text{Var}_F \rightarrow \mathcal{Z}_\text{Ab}$ (to the category of $\mathcal{Z}$-graded abelian groups) factors through $\mathcal{M}_F$, i.e., one has the commutative diagram of functors

$$
\begin{array}{ccc}
\text{Var}_F & \xrightarrow{\text{CH}} & \mathcal{Z}_\text{Ab} \\
\Gamma & \downarrow & \\
\mathcal{M}_F & \xrightarrow{R} & \mathcal{Z}_\text{Ab}
\end{array}
$$

where $\Gamma : f \mapsto \Gamma f$ is the graph and the functor $R$ is given by $R : (X, p) \mapsto \text{Im}(p^*)$, where $p^*$ is the composition

$$p^* : \text{CH}(X) \xrightarrow{\text{pr}_1^*} \text{CH}(X \times X) \xrightarrow{p} \text{CH}(X \times X) \xrightarrow{\text{pr}_2^*} \text{CH}(X).$$

3.3. Consider the morphism $(\text{id}, e) : \mathbb{P}^1 \times \{\text{pt}\} \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$. The image of the induced push-forward $(\text{id}, e)_*$ doesn’t depend on the choice of a point $e : \{\text{pt}\} \rightarrow \mathbb{P}^1$ and defines the projector in $\text{CH}^1(\mathbb{P}^1 \times \mathbb{P}^1)$ denoted by $p_1$. The motive $L = (\mathbb{P}^1, p_1)$ is called the Lefschetz motive. For a motive $M$ and an nonnegative integer $i$ we denote by $M(i) = M \otimes L^\otimes i$ its twist. Observe that

$$\text{Mor}((X, p)(i), (Y, q)(j)) = q \circ \text{CH}^{\dim X+i-j}(X, Y) \circ p.$$

3.4 (Grassmann Bundle Theorem). Let $X$ be a variety over $F$ and $\mathcal{E}$ be a vector bundle over $X$ of rank $n$. Then the motive of the Grassmann bundle $\text{Gr}(d, \mathcal{E})$ over $X$ is isomorphic to

$$\mathcal{M}(\text{Gr}(d, \mathcal{E})) \simeq \bigoplus_{\lambda} \mathcal{M}(X)(d(n-d) - |\lambda|),$$
where the sum is taken over all partitions \( \lambda = (\lambda_1, \ldots, \lambda_d) \) such that \( n - d \geq \lambda_1 \geq \ldots \geq \lambda_d \geq 0 \).

3.5 (Functors of Points). In sections 4.6 and 7 we use the functorial language, that is consider \( F \)-schemes as functors from the category of \( F \)-algebras to the category of sets. Fix a scheme \( X \). By an \( X \)-algebra we mean a pair \((R, x)\), where \( R \) is a \( F \)-algebra and \( x \) is an element of \( X(R) \). \( X \)-algebras form a category with obvious morphisms. Morphisms \( \varphi : Y \rightarrow X \) can be considered as functors from the category of \( X \)-algebras to the category of sets, by sending a pair \((R, x)\) to its preimage in \( Y(R) \).

3.6. Let \( X \) be a variety over \( F \). To any vector bundle \( F \) over \( X \) we can associate the Grassmann bundle \( Y = \text{Gr}(d, \mathcal{F}) \). Fix an \( X \)-algebra \((R, x)\). The value of the functor corresponding to \( \text{Gr}(d, \mathcal{F}) \) at \((R, x)\) is the set of direct summands of rank \( d \) of the projective \( R \)-module \( \mathcal{F}_x \otimes_F R \), where \( \mathcal{F}_x = \mathcal{F}(R, x) \).

4 Groups of type \( A_n \)

The goal of the present section is to prove Theorem 2.1 and Proposition 2.4. We use the notation of section 2.

4.1. Let \( G \) be an adjoint group of inner type \( A_n \) defined over a field \( F \). It is well known that \( G = \text{PGL}_1(A) \), where \( A \) is a central simple algebra of degree \( n + 1 \) and points of projective \( G \)-homogeneous varieties are flags of (right) ideals of \( A \)

\[
X(d_1, \ldots, d_k) = \{ I_1 \subset \ldots \subset I_k \subset A \mid \text{rdim } I_i = d_i \}.
\]

For convenience we set \( d_0 = 0, d_{k+1} = n + 1 \), \( I_0 = 0 \), \( I_{k+1} = A \).

4.2. The value of the functor of points corresponding to the variety \( X(d_1, \ldots, d_k) \) at a \( F \)-algebra \( R \) (see 3.5) equals the set of all flags \( I_1 \subset \ldots \subset I_k \) of right ideals of \( A_R = A \otimes_F R \) having the following properties (see [IK00, section 4])

- the injection of \( A_R \)-modules \( I_i \hookrightarrow A_R \) splits;
- \( \text{rdim } I_i = d_i \) (rdim means reduced rank over \( R \)).
4.3. On the scheme $X = X(d_1, \ldots, d_k)$ there are “tautological” vector bundles $J_i$, $i = 0, \ldots, k + 1$, of ranks $(n + 1)d_i$. The value of $J_i$ on an $X$-algebra $(R, x)$, where $x = (I_1, \ldots, I_k)$, is the ideal $I_i$ considered as a projective $R$-module. The bundle $J_i$ also has a structure of right $A_\Sigma X$-module, where $A_\Sigma X$ is the constant sheaf of algebras on $X$ determined by $A$.

For every $m \in \{1, \ldots, k\}$ there exists an obvious morphism

$$X(d_1, \ldots, d_k) \to X(d_1, \ldots, \hat{d}_m, \ldots, d_k)$$

$$(I_1, \ldots, I_k) \mapsto (I_1, \ldots, \hat{I}_m, \ldots, I_k)$$

that turns $X(d_1, \ldots, d_k)$ into a $X(d_1, \ldots, \hat{d}_m, \ldots, d_k)$-scheme.

4.4 Lemma. Denote $X(d_1, \ldots, d_k)$ by $Y$ and $X(d_1, \ldots, \hat{d}_m, \ldots, d_k)$ by $X$. Assume there exists a vector bundle $E$ over $X$ such that $A_\Sigma X \simeq \text{End}_{O_X}(E)$. Consider the vector bundle

$$F = J_{m+1}E/J_mE = J_{m+1}/J_{m-1} \otimes_{A_\Sigma X} E$$

of rank $d_{m+1} - d_{m-1}$. Then $Y$ as a scheme over $X$ can be identified with the Grassmann bundle $Z = \text{Gr}(d_m - d_{m-1}, F)$ over $X$.

Proof. We use essentially the same method as in [IK00, Proposition 4.3].

Fix an $X$-algebra $(R, x)$ where $x = (I_1, \ldots, \hat{I}_m, \ldots, I_k)$. The fiber of $Y$ over $x$, i.e., the value at $(R, x)$, can be identified with the set of all ideals $I_m$ satisfying the conditions [4.2] such that $I_{m-1} \subset I_m \subset I_{m+1}$. The fiber of $Z$ over $x$ is the set of all $R$-submodules $N$ of $F_y = F(R, y)$ such that the injection $N \hookrightarrow F_y$ splits and $\text{rk}_R N = d_m - d_{m-1}$.

We define a natural bijection between the fibers of $Y$ and $Z$ over $x$ as follows.

Consider the following mutually inverse bijections between the set of all right ideals of reduced dimension $r$ in $A_R$ (satisfying [4.2]) and the set of all direct summands of rank $r$ of the $R$-module $E_x$

$$\Phi : I \mapsto I E_x$$
$$\Psi : N \mapsto \text{Hom}_{R}(E_x, N) \subset \text{End}_R(E_x) \simeq A_R$$

Observe that these bijections preserve the respective inclusions of ideals and modules. So ideals of reduced dimension $d_m$ between $I_{m-1}$ and $I_m$ correspond to submodules of rank $d_m$ between $I_{m-1}E_x$ and $I_{m+1}E_x$, and, therefore, to submodules of rank $d_{m+1} - d_{m-1}$ in $I_{m+1}E_x/I_{m-1}E_x = F_x$. This gives the desired natural bijection on the fibers. \qed
4.5 Lemma. Suppose that $\gcd(\text{ind}(A), d_1, \ldots, d_k) = 1$. Then there exists a vector bundle $\mathcal{E}$ over $X = X(d_1, \ldots, d_k)$ of rank $n + 1$ such that $A_X \simeq \text{End}_{\mathcal{O}_X}(\mathcal{E})$.

Proof. We have to prove that the class $[A_X]$ in $\text{Br}(X)$ is trivial. Since $X$ is a regular Noetherian scheme the canonical map

$$\text{Br}(X) \to \text{Br}(F(X))$$

where $F(X)$ is the function field of $X$, is injective by [Gr65, 1.10] and [AG60, Theorem 7.2]. So it is enough to prove that $A \otimes F(X)$ splits. But the generic point of $X$ defines a flag of ideals of $A \otimes F(X)$ of reduced dimensions $d_1, \ldots, d_k$. Since the index $\text{ind}(A \otimes F(X))$ divides $d_1, \ldots, d_k$ and $\text{ind} A$, by the assumption of the lemma it must be equal to 1. So $A \otimes F(X)$ is split and this finishes the proof of the lemma.

4.6 Remark. In the case $d_1 = 1$ one can take $\mathcal{E} = \mathcal{J}_1$.

4.7 Remark. It can be shown using the Index Reduction Formula (see [MPW96]) that the condition on the gcd is necessary and sufficient for the central simple algebra $A_{F(X)}$ to be split.

We are now ready to finish the proof of 2.1

4.8 (Proof of Theorem 2.1). By Lemma 4.5 there exists a vector bundle $\mathcal{E}$ over variety $X = X(d_1, \ldots, d_m, \ldots, d_k)$ of rank $n + 1$ such that $A_X \simeq \text{End}_{\mathcal{O}_X}(\mathcal{E})$. By Lemma 4.4 we conclude that $Y = X(d_1, \ldots, d_k)$ is a Grassmann bundle over $X$. Now by 3.4 we obtain the isomorphism of 2.1.

4.9 Remark. Note that the assumption of Theorem 2.1 on the reduced dimensions $d_1, \ldots, d_k$ is essential. Indeed, suppose the Theorem holds for any twisted flag variety. Consider the flag $X = X(1, d)$ with $\gcd(\text{ind}(A), d) > 1$. Then we have an isomorphism of motives

$$\mathcal{M}(X) \simeq \bigoplus_{i=0}^{d-1} \mathcal{M}(\text{SB}_d(A))(i)$$

which appears after applying Theorem to the flags $X(1, d)$ and $X(d)$. Consider the group $\text{CH}_0(X) = \text{Mor}_{\mathcal{M}}(\mathcal{M}(pt), \mathcal{M}(X))$. The isomorphism above induces the isomorphism of groups

$$\text{Coker}(\text{CH}_0(X) \overset{\text{res}}{\to} \text{CH}_0(X_s)) \cong \text{Coker}(\text{CH}_0(\text{SB}_d(A)) \overset{\text{res}}{\to} \text{CH}_0(\text{Gr}(d, n + 1)))$$

$$\cong \mathbb{Z}/(\frac{\text{ind}(A)}{\gcd(\text{ind}(A), d)}\mathbb{Z})$$
where res is the pull-back induced by the scalar extension $F_s/F$ and the last isomorphism follows by [Bl91, Theorem 3]. On the other hand, applying Theorem 2.1 to the flags $X(1,d)$ and $X(1)$ we obtain the isomorphism

$$M(X) \cong \bigoplus_{\lambda} M(SB(A))((n + 1 - d)(d - 1) - |\lambda|)$$

which induces the isomorphism of groups

$$\text{Coker}(CH_0(X) \xrightarrow{\text{res}} CH_0(X_s)) \cong \text{Coker}(CH_0(SB(A)) \xrightarrow{\text{res}} CH_0(\mathbb{P}^n)) \cong \mathbb{Z}/\text{ind}(A)\mathbb{Z},$$

that leads to a contradiction.

We now prove Proposition 2.4.

4.10 (Proof of Proposition 2.4). Let $G = \text{PGL}_1(A)$ and let $\mathcal{M}(G,R)$ be the symmetric tensor category of motives of $G$-homogeneous varieties with coefficients in a ring $R$ for which the Krull-Schmidt theorem holds. It is the case, e.g., when $R$ is a field or, more general, a discrete valuation ring (see [CM04, Theorem 9.6]).

Consider the $G$-homogeneous variety $X(1,d)$, $1 < d < n$. Apply Theorem 2.1 to the sequences of flags $X(1,d)$, $X(d)$ and $X(1,d)$, $X(1)$. We obtain two isomorphisms in $\mathcal{M}(G,R)$

$$\bigoplus_{i=0}^{d-1} \mathcal{M}(SB_d(A))(i) \cong \mathcal{M}(X) \cong \bigoplus_{\lambda} \mathcal{M}(SB(A))((n + 1 - d)(d - 1) - |\lambda|), \quad (*)$$

where the sum on the right hand side is taken over all partitions $\lambda = (\lambda_1, \ldots, \lambda_{d-1})$ such that $n+1-d \geq \lambda_1 \geq \ldots \geq \lambda_{d-1} \geq 0$. Since Krull-Schmidt Theorem holds in $\mathcal{M}(G,R)$, the motive $SB(A)$ has a unique decomposition into the direct sum of indecomposable objects $H_i$, $i \in \mathcal{I}$, and their twists

$$\mathcal{M}(SB(A)) \cong \bigoplus_{i \in \mathcal{I}} (\oplus_{j \in \mathcal{J}_i} H_i(j)).$$

Consider the subcategory $\mathcal{M}(G,R)_\mathcal{I}$ additively generated by the motives $H_i$, $i \in \mathcal{I}$, and their twists. The abelian group of isomorphism classes of objects of this category can be equipped with a structure of a free module
over the polynomial ring $R[z]$. Namely, multiplication by $z$ is given by the twist. Clearly, the classes $[H_i]$, $i \in I$, form the basis of this $R[z]$-module.

By (*) we have $\mathcal{M}(\text{SB}_d(A)) \in \mathcal{M}(G, R)_I$ and the isomorphisms (*) can be rewritten as

$$\frac{z^d - 1}{z - 1} [\text{SB}_d(A)] = \frac{\varphi_n(z)}{\varphi_{d-1}(z)\varphi_{n+1-d}(z)} [\text{SB}(A)] = \frac{z^d - 1}{z - 1} \frac{\varphi_n(z)}{\varphi_d(z)\varphi_{n+1-d}(z)} [\text{SB}(A)]$$

where $\varphi_n(z) = \prod_{k=2}^n \frac{z^k - 1}{z - 1}$. This immediately implies the equality

$$[\text{SB}_d(A)] = \frac{\varphi_n(z)}{\varphi_d(z)\varphi_{n+1-d}(z)} [\text{SB}(A)],$$

i.e., the isomorphism in $\mathcal{M}(G, R)_I$ between $\mathcal{M}(\text{SB}_d(A))$ and the respective sum of twists of $\mathcal{M}(\text{SB}(A))$. This finishes the proof of the proposition.

5 Motivic decomposition of $\text{SB}_2(A)$

This section is devoted to the proof of Theorem 2.5.

First, we need to recall some properties of rational cycles on projective homogeneous varieties.

5.1. Let $G$ be a split linear algebraic group over $F$. Let $X$ be a projective $G$-homogeneous variety, i.e., $X = G/P$, where $P$ is a parabolic subgroup of $G$. The abelian group structure of $\text{CH}(X)$, as well as its ring structure, is well-known. Namely, $X$ has a cellular filtration and the generators of Chow groups of the bases of this filtration correspond to the free additive generators of $\text{CH}(X)$ (see [Ka01]). Note that the product of two projective homogeneous varieties $X \times Y$ has a cellular filtration as well, and $\text{CH}^*(X \times Y) \cong \text{CH}^*(X) \otimes \text{CH}^*(Y)$ as graded rings. The correspondence product of two cycles $\alpha = f_\alpha \times g_\alpha \in \text{CH}(X \times Y)$ and $\beta = f_\beta \times g_\beta \in \text{CH}(Y \times X)$ is given by (cf. [Bo03, Lem. 5])

$$(f_\beta \times g_\beta) \circ (f_\alpha \times g_\alpha) = \text{deg}(g_\alpha \cdot f_\beta)(f_\alpha \times g_\beta),$$

where $\text{deg} : \text{CH}(Y) \to \text{CH}([pt]) = \mathbb{Z}$ is the degree map.
5.2. Let $X$ be a projective variety of dimension $n$ over a field $F$. Let $F_s$ be the separable closure of the field $F$. Consider the scalar extension $X_s = X \times_F F_s$. We say a cycle $J \in \text{CH}(X_s)$ is rational if it lies in the image of the pull-back homomorphism $\text{CH}(X) \to \text{CH}(X_s)$. For instance, there is an obvious rational cycle $\Delta_{X_s}$ on $\text{CH}^n(X_s \times X_s)$ that is given by the diagonal class. Clearly, linear combinations, intersections and correspondence products of rational cycles are rational.

5.3. We will use the following fact (see [CGM05, Cor. 8.3]) that follows from the Rost Nilpotence Theorem. Let $p_s$ be a non-trivial rational projector in $\text{CH}^n(X_s \times X_s)$, i.e., $p_s \circ p_s = p_s$. Then there exists a non-trivial projector $p$ on $\text{CH}^n(X \times X)$ such that $p \times_F F_s = p_s$. Hence, the existence of a non-trivial rational projector $p_s$ on $\text{CH}^n(X_s \times X_s)$ gives rise to the decomposition of the Chow motive of $X$

$$M(X) \cong (X, p) \oplus (X, \Delta_X - p)$$

Our goal is to find such a projector in the case $X = SB_d(A)$.

5.4. An isomorphism between the twisted motives $(X, p)(m)$ and $(Y, q)(l)$ is given by correspondences $j_1 \in \text{CH}^{\dim X - l + m}(X \times Y)$ and $j_2 \in \text{CH}^{\dim Y - m + l}(Y \times X)$ such that $q \circ j_1 = j_1 \circ p$, $p \circ j_2 = j_2 \circ q$ and $j_1 \circ j_2 = q$, $j_2 \circ j_1 = p$. If $X$ and $Y$ lie in the category $\mathcal{M}(G, \mathbb{Z})$ then by the Rost nilpotence theorem (see [CM04, Theorem 8.2] and [CGM05, Corollary 8.4]) it suffices to give a rational $j_1$ and some $j_2$ satisfying these conditions over separable closure (note that $j_2$ will automatically be rational).

We now recall some properties of Grassmann varieties and describe their Chow rings.

5.5. Consider the Grassmann variety $\text{Gr}(d, n + 1)$, $1 \leq d \leq n$, of $d$-planes in the $(n + 1)$-dimensional affine space. It has the dimension $d(n + 1 - d)$. A twisted form of it is a generalized Severi-Brauer variety $SB_d(A)$, where $A$ is a central simple algebra of degree $n + 1$. For any two integers $d$ and $d'$, $1 \leq d, d' \leq n$, there is the fiber product diagram

$$\begin{array}{c}
\text{Gr}(d, n + 1) \times \text{Gr}(d', n + 1) \xrightarrow{\text{Seg}} \text{Gr}(dd', (n + 1)^2) \\
\downarrow \\
\text{SB}_d(A) \times \text{SB}_{d'}(A^{\text{op}}) \xrightarrow{\text{Seg}} \text{SB}_{dd'}(A \otimes A^{\text{op}})
\end{array}$$

(1)
where the horizontal arrows are Segre embeddings given by the tensor product of ideals (resp. linear subspaces) and the vertical arrows are canonical maps induced by the scalar extension \( F_s/F \).

5.6. The diagram (1) induces the commutative diagram of rings

\[
\begin{array}{ccc}
\text{CH}(\text{Gr}(d, n+1) \times \text{Gr}(d', n+1)) & \xrightarrow{\text{Seg}^*} & \text{CH}(\text{Gr}(dd', (n+1)^2)) \\
\uparrow\text{res} & & \uparrow\text{res} \\
\text{CH}(\text{SB}_d(A) \times \text{SB}_{d'}(A^{\text{op}})) & \xrightarrow{\text{Seg}^*} & \text{CH}(\text{SB}_{dd'}(A \otimes A^{\text{op}}))
\end{array}
\]

where all maps are the induced pull-backs. Observe that the right vertical arrow is an isomorphism since \( A \otimes A^{\text{op}} \) splits. Consider a vector bundle \( E \) over \( \text{Gr}(dd', (n+1)^2) \). It is easy to see that the pull-back of the total Chern class \( \text{Seg}^*(c(E)) \) is a rational cycle on \( \text{CH}(\text{Gr}(d, n+1) \times \text{Gr}(d', n+1)) = \text{CH}(\text{Gr}(d, n+1)) \otimes \text{CH}(\text{Gr}(d', n+1)) \). In particular, if \( E = \tau_{dd'} \) is the tautological bundle of \( \text{Gr}(dd', (n+1)^2) \) we obtain the following

5.7 Lemma. The total Chern class \( c(\text{pr}_1^*\tau_d \otimes \text{pr}_2^*\tau_{d'}) \) of the tensor product of the pull-backs (induced by the projection maps) of the tautological bundles \( \tau_d \) and \( \tau_{d'} \) of \( \text{Gr}(d, n+1) \) and \( \text{Gr}(d', n+1) \) respectively is rational.

From now on we restrict ourselves to the case \( n = 4, d = 2 \) and \( d' = 1 \), i.e., to the Grassmannian \( \text{Gr}(2, 5) \) and the projective space \( \mathbb{P}^4 = \text{Gr}(1, 5) \).

5.8. We describe the generators and relations of the Chow ring \( \text{CH}(\text{Gr}(2, 5)) \) following [Fu98, section 14.7]. Set \( \sigma_m = c_m(Q), m = 1, 2, 3 \), where \( Q = \mathcal{O}^5/\tau_2 \) is the universal quotient bundle of rank 3 over \( \text{Gr}(2, 5) \). It is known that the elements \( \sigma_m \) generate the Chow ring \( \text{CH}(\text{Gr}(2, 5)) \). More precisely, as an abelian group this ring is generated by the Schubert cycles \( \Delta_\lambda(\sigma) \) that are parameterized by all partitions \( \lambda = (\lambda_1, \lambda_2) \) such that \( 3 \geq \lambda_1 \geq \lambda_2 \geq 0 \). In particular, \( \sigma_m = \Delta_{(m,0)}, m = 1, 2, 3 \). For other generators we set the following notation \( g_2 = \Delta_{(1,1)}, g_3 = \Delta_{(2,1)}, h_4 = \Delta_{(3,1)}, g_4 = \Delta_{(2,2)}, g_5 = \Delta_{(3,2)}, pt = \Delta_{(3,3)} \). These generators corresponds to the vertices of the Hasse diagram of \( \text{Gr}(2, 5) \) (see [Hi82])

\[
\begin{array}{ccc}
\sigma_3 & & \sigma_1 \\
\uparrow & & \uparrow \\
\sigma_2 & & 1 \\
\downarrow & & \downarrow \\
g_4 & & g_2 \\
\downarrow & & \downarrow \\
pt & & g_5 \\
\end{array}
\]
The multiplication rules can be determined using Pieri formulae

\[ \Delta_\lambda \cdot \sigma_m = \sum_\mu \Delta_\mu, \]

where the sum is taken over all partitions \( \mu = (\mu_1, \mu_2) \) such that \( 3 \geq \mu_1 \geq \lambda_1 \geq \mu_2 \geq \lambda_2 \geq 0 \).

5.9. Consider the tautological bundle \( \tau_2 \) of the Grassmannian \( \text{Gr}(2, 5) \). Its total Chern class is

\[ c(\tau_2) = c(Q)^{-1} = \frac{1}{1 + \sigma_1 + \sigma_2 + \sigma_3} = 1 - \sigma_1 - \sigma_2 + \sigma_1^2 + \ldots \]

where the rest consists of the summands of degree greater than 2. Hence, we obtain \( c_1(\tau_2) = -\sigma_1 \) and \( c_2(\tau_2) = -\sigma_2 + \sigma_1^2 = g_2 \).

5.10. The Chow ring of the projective space \( \mathbb{P}^4 \) can be identified with the factor ring \( \mathbb{Z}[H]/(H^5) \), where \( H = c_1(\mathcal{O}(1)) \) is the class of a hyperplane section. Thus, the first Chern class of the tautological bundle of \( \mathbb{P}^4 \) equals to \( c_1(\mathcal{T}_1) = c_1(\mathcal{O}(-1)) = -H \).

We are now ready to prove Theorem 2.5.

5.11 (Proof of Theorem 2.5). By Lemma 5.7 we obtain the following rational cycles in the group \( \text{CH}^*(\text{Gr}(2, 5) \times \mathbb{P}^4) \)

\[ r = c_1(\text{pr}_1^*(\tau_2) \otimes \text{pr}_2^*(\tau_1)) = c_1(\text{pr}_1^*(\tau_2)) + 2c_1(\text{pr}_2^*(\tau_1)) = -\sigma_1 \times 1 - 2(1 \times H), \]

\[ \rho = c_2(\text{pr}_1^*(\tau_2) \otimes \text{pr}_2^*(\tau_1)) = c_2(\text{pr}_1^*(\tau_2)) + c_1(\text{pr}_1^*(\tau_2))c_1(\text{pr}_2^*(\tau_1)) + c_1(\text{pr}_2^*(\tau_1))^2 \]

\[ = g_2 \times 1 + \sigma_1 \times H + 1 \times H^2 \]

For two cycles \( x \) and \( y \) we shall write \( x \equiv y \) if there exists a cycle \( z \) such that \( x - y = 5z \). Note that \( \equiv \) is an equivalence relation that preserves rationality of cycles. Then the following cycles are rational

\[ \rho^2 \equiv 1 \times H^4 + 2\sigma_1 \times H^3 + (\sigma_2 + 3g_2) \times H^2 + 2g_3 \times H + g_4 \times 1, \]

\[ \rho^3 \equiv (3\sigma_2 + g_2) \times H^4 + (\sigma_3 + 3g_3) \times H^3 + (g_4 + 3h_4) \times H^2 + 3g_5 \times H + pt \times 1. \]

Consider the composition

\[ (\rho^2)^4 \circ \rho^3 \equiv (3\sigma_2 + g_2) \times g_4 + (2\sigma_3 + g_3) \times g_3 \]

\[ + (g_4 + 3h_4) \times (\sigma_2 - 2g_2) + g_5 \times \sigma_1 + pt \times 1. \]
Note that the right-hand side is a rational projector (over \( \mathbb{Z} \)) and, therefore, by Rost nilpotence theorem (see \cite[Corollary 8.3]{CGM05}) has a form \( p \times F \) where \( p \) is a projector in \( \text{End}(\mathcal{M}(\text{SB}_2(A))) \). The latter determines an object \((\text{SB}_2(A), p)\) in the category of motives (actually in \( \mathcal{M}(G, \mathbb{Z}) \)) which we denote by \( \mathcal{H} \).

Set \( q = \Delta_{\text{SB}_2(A)} - p \). We then show that

\[
(\mathcal{M}(\text{SB}_2(A)), q) \simeq (\mathcal{M}(\text{SB}_2(A)), p^t) \simeq \mathcal{H}(2),
\]

which gives the decomposition \( \mathcal{M}(\text{SB}_2(A)) \simeq \mathcal{H} \oplus \mathcal{H}(2) \).

Observe that an isomorphism \((\text{SB}_2(A), q) \simeq (\text{SB}_2(A), p^t)\) is given by the two mutually inverse motivic isomorphisms \( p^t \circ q_s \) and \( q_s \circ p^t \) over \( F \), which are rational. An isomorphism \( \mathcal{H}(2) \simeq (\mathcal{M}(\text{SB}_2(A)), p^t) \) is given by the following two cycles

\[
j_1 = (3\sigma_2 + g_2) \times pt - (2\sigma_3 + g_3) \times g_5
+ (g_4 + 3h_4) \times (g_4 + 3h_4) - g_5 \times (2\sigma_3 + g_3) + pt \times (3\sigma_2 + g_2),
\]

\[
j_2 = 1 \times g_4 - \sigma_1 \times g_3 + (\sigma_2 - 2g_2) \times (\sigma_2 - 2g_2) - g_3 \times \sigma_1 + g_4 \times 1.
\]

Note that \( j_1 \) is rational, since \( j_1 \equiv (1 \times (3\sigma_2 + g_2))p \), and \( 1 \times (3\sigma_2 + g_2) \equiv 3(\rho + \tau^2)^t \circ \rho^2 \) is rational.

Since \( A \) is a division algebra of degree 5, there is a division algebra \( B \) of degree 5 Brauer-equivalent to the tensor square \( A \otimes A \). We claim that \( \mathcal{H} \simeq \mathcal{M}(\text{SB}(B)) \). By the exact sequence (see \cite[Remark 7.17]{Ka00})

\[
\text{CH}^1(\text{SB}(A^{\text{op}}) \times \text{SB}(B)) \xrightarrow{\text{res}_{F_s/F}} \text{CH}^1(\mathbb{P}^4 \times \mathbb{P}^4) \xrightarrow{H \times 1 - [A^{\text{op}}]} \text{Br}(F) \xrightarrow{1 \times H - [B]} \text{Br}(F)
\]

the following cycle in \( \text{CH}^1(\mathbb{P}^4 \times \mathbb{P}^4) \) is rational

\[
u = 2H \times 1 + 1 \times H.
\]

Therefore the cycles

\[
\alpha = pt \times 1 + g_5 \times H - (g_4 + 3h_4) \times H^2 - (g_3 + 2\sigma_3) \times H^3 + (3\sigma_2 + g_2) \times H^4
\equiv u^4 \circ \rho^3,
\]

\[
\beta = 1 \times g_4 - H \times g_3 - H^2 \times (\sigma_2 - 2g_2) + H^3 \times \sigma_1 + H^4 \times 1 \equiv (\rho^2)^t \circ (u^4)^t
\]

are rational. A direct computation shows that \( \alpha \circ \beta = \Delta_{\mathbb{P}^4} \) and \( \beta \circ \alpha = p_s \).

Therefore, by Rost nilpotence theorem \( \mathcal{H} \simeq \mathcal{M}(\text{SB}(B)) \). This finishes the proof of the theorem.

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6 Groups of types $B_n$ and $C_n$

The goal of the present section is to prove Theorems 2.9 and 2.11.

6.1. Let $G$ be an adjoint group of type $B_n$. From now on we suppose that the characteristic of $F$ is not 2. It is known that $G = O^+(V,q)$, where $(V,q)$ is a regular quadratic space of dimension $2n + 1$ and projective $G$-homogeneous varieties can be described as flags of $q$-totally isotropic subspaces

$$X(d_1,\ldots,d_k) = \{V_1 \subset \ldots \subset V_k \subset V \mid \dim V_i = d_i\}.$$ 

6.2. The value of the functor corresponding to the variety $X(d_1,\ldots,d_k)$ at a $F$-algebra $R$ equals the set of all flags $V_1 \subset \ldots \subset V_k$, where $V_i$ is a $q_R$-totally isotropic direct summand of $V_R$ of rank $d_i$.

For convenience we set $d_0 = 0$, $V_0 = 0$.

6.3. On the scheme $X = X(d_1,\ldots,d_k)$ there are “tautological” vector bundles $V_i$ of ranks $d_i$. The value of $V_i$ on an $X$-algebra $(R,x)$ is $V_i$, where $x = (V_1,\ldots,V_k)$. For every $m$ there exists an obvious morphism

$$X(d_1,\ldots,d_k) \rightarrow X(d_1,\ldots,\hat{d}_m,\ldots,d_k)$$

$$(V_1,\ldots,V_k) \mapsto (V_1,\ldots,\hat{V}_m,\ldots,V_k)$$

which makes $X(d_1,\ldots,d_k)$ into a $X(d_1,\ldots,\hat{d}_m,\ldots,d_k)$-scheme.

6.4 Lemma. Denote the variety $X(d_1,\ldots,d_k)$ by $Y$ and $X(d_1,\ldots,\hat{d}_m,\ldots,d_k)$ by $X$. Suppose that $m < k$. Then $Y$ as a scheme over $X$ can be identified with the Grassmann bundle $Z = \text{Gr}(d_m - d_{m-1}, V_{m+1}/V_{m-1})$ over $X$.

Proof. Fix an $X$-algebra $(R,x)$, where $x = (V_1,\ldots,\hat{V}_m,\ldots,V_k)$. We define a natural bijection between the fibers over the point $x$ of $Y$ and $Z$ as follows. The fiber of $Y$ over $x$ can be identified with the set of all direct summands $V_m$ of $V_R$ of rank $d_m$ such that $V_{m-1} \subset V_m \subset V_{m+1}$ (note that $V_m$ is automatically $q_R$-isotropic since $V_{m+1}$ is so). This fiber is clearly isomorphic to the fiber of $Z$ over $x$ which is the set of all direct summands of $(V_{m+1}/V_{m-1})_x = V_{m+1}/V_{m-1}$ of rank $d_m$. 

6.5 (Proof of Theorem 2.9). Apply Lemma 6.4 to the varieties $Y = X(d_1,\ldots,d_k)$ and $X = X(d_1,\ldots,\hat{d}_m,\ldots,d_k)$. We obtain that $Y$ is a Grassmann bundle over $X$. To finish the proof apply 3.4.
6.6. Let \( G \) be an adjoint group of type \( C_n \) over \( F \). It is known that \( G = \text{Aut}(A, \sigma) \), where \( A \) is a central simple algebra of degree \( 2n \) with an involution \( \sigma \) of symplectic type on \( A \), and projective \( G \)-homogeneous varieties can be described as flags of (right) ideals of \( A \)

\[
X(d_1, \ldots, d_k) = \{ I_1 \subset \ldots \subset I_k \subset A \mid I_i \subseteq I_i^\perp, \text{rdim } I_i = d_i \}.
\]

Here \( I_i^\perp = \{ x \in A \mid \sigma(x)I = 0 \} \) is a right ideal of reduced dimension \( 2n - \text{rdim } I \).

6.7. The value of the functor corresponding to the variety \( X(d_1, \ldots, d_k) \) at a \( F \)-algebra \( R \) equals to the set of all flags \( I_1 \subset \ldots \subset I_k \) of right ideals of \( A_R = A \otimes_F R \) having the following properties

- the injection of \( A_R \)-modules \( I_i \hookrightarrow A_R \) splits;
- \( I_i \subseteq I_i^\perp \);
- \( \text{rdim } I_i = d_i \).

For convenience we set \( I_0 = 0 \).

6.8. On the scheme \( X = X(d_1, \ldots, d_k) \) there are “tautological” vector bundles \( J_i \) of ranks \( 2d_i \) and their “orthogonal complements” \( J_i^\perp \) of rank \( 2n(2n - d_i) \). The value of \( J_i \) (resp. \( J_i^\perp \)) on an \( X \)-algebra \((R, x)\), where \( x = (I_1, \ldots, I_k) \), is \( I_i \) (resp. \( I_i^\perp \)) considered as a projective \( R \)-module. The bundles \( J_i \) and \( J_i^\perp \) also have structures of right \( A_X \)-modules, where \( A_X \) is a constant sheaf of algebras on \( X \) determined by \( A \). There exists an obvious morphism

\[
X(d_1, \ldots, d_k) \to X(d_1, \ldots, d_{k-1})
\]

\[
(I_1, \ldots, I_k) \mapsto (I_1, \ldots, I_{k-1}),
\]

which makes \( X(d_1, \ldots, d_k) \) into a \( X(d_1, \ldots, d_{k-1}) \)-scheme.

6.9 Lemma. Denote \( X(d_1, \ldots, d_k) \) by \( Y \) and \( X(d_1, \ldots, d_{k-1}) \) by \( X \). Suppose that \( d_k = d_{k-1} + 1 \) and there exists a vector bundle \( E \) over \( X \) such that \( A_X \simeq \text{End}_{\mathcal{O}_X}(E) \). Consider the vector bundle

\[
F = J_{k-1}^\perp E / J_{k-1} E = J_{k-1}^\perp / J_{k-1} \otimes_{A_X} E
\]

of rank \( 2(n - d_{k-1}) \). Then \( Y \) as a scheme over \( X \) can be identified with the projective bundle \( Z = \mathbb{P}(F) = \text{Gr}(1, F) \) over \( X \).
Proof. Fix an $X$-algebra $(R, x)$, where $x = (I_1, \ldots, I_{k-1})$. We define a natural bijection between the fibers over the point $x$ of $Y$ and $Z$. The fiber of $Y$ can be identified with the set of all ideals $I_k$ containing $I_{k-1}$ and satisfying the conditions 6.7. The fiber of $Z$ is the set of all direct summands of $F_x = \mathcal{F}(R, x)$ of rank 1.

The involution $\sigma$ induces an isomorphism $h : \mathcal{E}_x \otimes L \to \mathcal{E}_x^*$ for some invertible $R$-module $L$ (see [Kn91, Lemma III.8.2.2]) such that

$$\sigma(f) \otimes 1 = h^{-1} f^* h \text{ for all } f \in A$$

$$h^* \text{ can } \otimes 1 = -h$$

where $\text{can} : \mathcal{E}_x \to \mathcal{E}_x^{**}$ is the canonical isomorphism.

Let $U_1$ and $U_2$ be direct summands of $\mathcal{E}_x$. We write $U_2 \subseteq U_1^\perp$ if $h(u \otimes l)(v) = 0$ for all $u \in U_1$, $v \in U_2$, $l \in L$. We call a direct summand $U$ of $\mathcal{E}_x$ totally isotropic if $U \subseteq U^\perp$. Note that any direct summand of rank 1 is totally isotropic (it can be proved easily using localization).

Define $\Phi$ and $\Psi$ as in the proof of Theorem 4.4. Direct computations show that $I_1 \subseteq I_2^\perp$ if and only if $\Phi(I_1) \subseteq \Phi(I_2)^\perp$.

So the fiber of $Y$ over $x$ is naturally isomorphic to the set of all totally isotropic direct summands $U_k$ of $\mathcal{E}_x$ of rank $d_k$ containing $U_{k-1} = \Phi(I_k)$. One can represent $U_k$ as the direct sum $U_{k-1} \oplus U$ where $U$ is a direct summand of rank 1 (since $d_k = d_{k-1} + 1$). This $U$ is totally isotropic and, therefore, $U_k$ is totally isotropic if and only if $U \subseteq U_{k-1}^\perp$. Hence the set of all $U_{k-1}$ is naturally isomorphic to the set of all direct summands of $\Phi(I_{k-1}^\perp)$ of rank $d_k$ containing $\Phi(I_{k-1})$. The latter can be identified with $\mathbb{P}(\mathcal{F}_x)$. This finishes the proof of the lemma.

We are now ready to prove Theorem 2.11.

6.10 (Proof of Theorem 2.11). Consider the flag varieties $Y = X(d_1, \ldots, d_k)$ and $X(d_1, \ldots, d_{k-1})$. Since $\text{ind } A = 2^r$ for some $r$ and there is an odd $d_i$, we have $\gcd(\text{ind}(A), d_1, \ldots, d_{k-1}) = 1$. By Lemma 4.5 there exists a bundle $\mathcal{E}$ over $X$ such that $A_X = \text{End}_{\mathcal{O}_X}(\mathcal{E})$. Applying Lemma 6.9 to the varieties $X$, $Y$ and the bundle $\mathcal{E}$, we obtain that $Y$ is a projective bundle over $X$. Now we use 3.4 to finish the proof.

7 Groups of types $G_2$ and $F_4$

This section is devoted to proofs of Theorems 2.13 and 2.14.
7.1. Let $G$ be a group of type $G_2$. We suppose that characteristic of $F$ is not 2. It is known that $G = \text{Aut}(C)$ where $C$ is a Cayley algebra over $F$. By $i$-space where $i = 1, 2$ we mean an $i$-dimensional subspace $V_i$ of $C$ such that $uv = 0$ for every $u, v \in V_i$.

The only flag variety corresponding to a non-maximal parabolic is the full flag variety $X(1, 2)$ which is described as follows (see [Bo03]):

$$X(1, 2) = \{ V_1 \subset V_2 \mid V_i \text{ is an } i\text{-subspace of } C \}.$$ 

Similarly one can describe the homogeneous flag variety corresponding to the maximal parabolic 

$$X(2) = \{ V \mid V \text{ is a 2-subspace of } C \}.$$ 

7.2. Let $R$ be a $F$-algebra. By an $i$-submodule in $C_R = C \otimes_F R$ we mean a direct summand $V_i$ of $C_R$ of rank $i$ such that $uv = 0$ for every two elements $u, v \in V_i$. The value of the functor corresponding to the variety $X(1, 2)$ (respectively $X(2)$) at a $F$-algebra $R$ equals the set of all flags $V_1 \subset V_2$ (respectively submodules $V_2$) where $V_i$ is an $i$-submodule of $C_R$.

7.3. On the scheme $X = X(2)$ there is a “tautological” vector bundle $\mathcal{V}$ of rank 2. The value of $\mathcal{V}$ on an $X$-algebra $(R, x)$ is $V$, where $x = V$.

There exists an obvious morphism

$$X(1, 2) \to X(2)$$

$$(V_1, V_2) \mapsto V_2$$

which makes $X(1, 2)$ into a $X(2)$-scheme.

7.4 Lemma. $X(1, 2)$ as a scheme over $X(2)$ can be identified with the projective bundle $\mathbb{P}(\mathcal{V}) = \text{Gr}(1, \mathcal{V})$ over $X(2)$.

Proof. The proof goes as in $B_2\mu$-case (note that if $V_2$ is a 2-submodule then each of its direct summands of rank 1 is a 1-submodule).

7.5 (Proof of Theorem 2.13). Apply Lemma 7.4 and 3.4.

7.6. Let $G$ be a group of type $F_4$. We suppose that characteristic of $F$ is not 2, 3. It is known that $G = \text{Aut}(J)$ where $J$ is an exceptional Jordan algebra of dimension 27 over $F$. Set $I = \{1, 2, 3, 6\}$. By an $i$-space where $i \in I$ we
mean an $i$-dimensional subspace $V_i$ of $J$ such that every $u, v \in V_i$ satisfy the following condition:

$$\text{Tr}(u) = 0, \ u \times v = 0, \ \text{and if } i < 6 \text{ then } u(va) = v(ua) \text{ for all } a \in J.$$  

It is known that projective $G$-homogeneous varieties are parameterized by sequences of numbers $d_1 < \ldots < d_k$ from $I$ and can be described as follows:

$$X(d_1, \ldots, d_k) = \{ V_1 \subset \ldots \subset V_k \mid V_i \text{ is a } d_i\text{-subspace of } A \}.$$  

7.7. Let $R$ be a $F$-algebra. By an $i$-submodule in $J_R = J \otimes_F R$ we mean a direct summand $V_i$ of rank $i$ such that every two elements $u, v \in V_i$ satisfy the conditions above. The value of the functor corresponding to the variety $X(d_1, \ldots, d_k)$ at a $F$-algebra $R$ equals the set of all flags $V_1 \subset \ldots \subset V_k$ where $V_i$ is a $d_i$-submodule of $J_R$.

For convenience we set $d_0 = 0, \ V_0 = 0$.

7.8. On the scheme $X = X(d_1, \ldots, d_k)$ there are “tautological” vector bundles $V_i$ of rank $d_i$. The value of $V_i$ on an $X$-algebra $(R, x)$ is $V_i$, where $x = (V_1, \ldots, V_k)$.

There exists an obvious morphism

$$X(d_1, \ldots, d_k) \to X(d_1, \ldots, \hat{d}_m, \ldots, d_k)$$

$$(V_1, \ldots, V_k) \mapsto (V_1, \ldots, \hat{V}_m, \ldots, V_k)$$

which makes $X(d_1, \ldots, d_k)$ into a $X(d_1, \ldots, \hat{d}_m, \ldots, d_k)$-scheme.

7.9 Lemma. Denote $X(d_1, \ldots, d_k)$ by $Y$ and $X(d_1, \ldots, \hat{d}_m, \ldots, d_k)$ by $X$. Suppose that $m < k$ and either $d_{m+1} < 6$ or $d_m = 1$. Then $Y$ as a scheme over $X$ can be identified with the Grassmann bundle $Z = \text{Gr}(d_m - d_{m-1}, V_{m+1}/V_{m-1})$ over $X$.

Proof. The proof goes as in $B_n$-case (note that under our restrictions if $V_{m+1}$ is a $d_{m+1}$-submodule then each of its direct summands of rank $d_m$ is a $d_m$-submodule).

7.10 (Proof of Theorem 2.14). Apply Lemma 7.9 to the varieties $Y = X(d_1, \ldots, d_k)$ and $X = X(d_1, \ldots, \hat{d}_m, \ldots, d_k)$. We obtain that $Y$ is a Grassmann bundle over $X$. To finish the proof apply 3.4.
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References


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