Logic for exact real arithmetic

Helmut Schwichtenberg
Joint work with Ulrich Berger (Swansea), Nils Köpp (LMU), Kenyi Miyamoto (Innsbruck), Hideki Tsuiki (Kyoto) and Franziskus Wiesnet (LMU)

Mathematisches Institut, LMU, München

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Exact real numbers

can be given in different formats:

- Cauchy sequences (of rationals, with Cauchy modulus).
- Infinite sequences ("streams") of signed digits $\{-1, 0, 1\}$, or
- $\{-1, 1, \bot\}$ with at most one $\bot$ ("undefined"): Gray code.

Want formally verified algorithms on reals given as streams.

- Consider formal proofs M and apply realizability to extract their computational content.
- Switch between different formats of reals by decoration. Example:

$$\forall_x A \quad \mapsto \quad \forall_x^{\mathrm{nc}}(x \in {}^{\mathrm{co}}I \to A)).$$

- Computational content of $x \in {}^{\mathrm{co}}I$ is a stream representing x.

A real number can be represented as a Cauchy sequence $(a_n)_n$ of rationals together with a Cauchy modulus M satisfying

$$|a_n - a_m| \leq \frac{1}{2^p} \quad \text{for } n, m \geq M(p).$$

Arithmetical operations on real numbers x, y are defined by

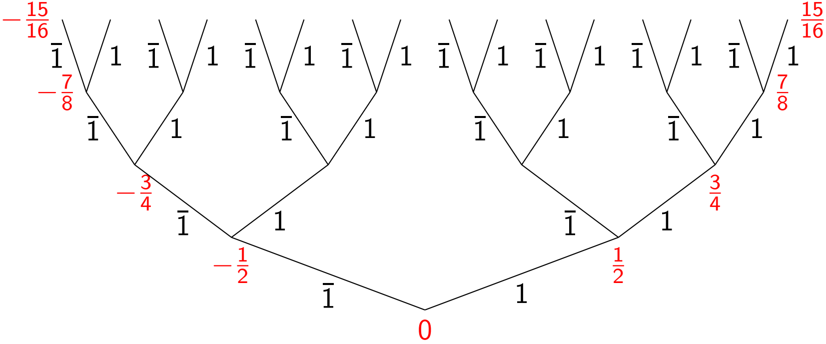
	c_n	$L(p)$
$x + y$	$a_n + b_n$	$\max\big(M(p+1), N(p+1)\big)$
$-x$	$-a_n$	$M(p)$
$\lvert x\rvert$	$\lvert a_n\rvert$	$M(p)$
$x \cdot y$	$a_n \cdot b_n$	$\max\big(M(p+1+p_y), N(p+1+p_x)\big)$
$\frac{1}{x}$ for $\lvert x\rvert \in_q \mathbb{R}^+$	$\begin{cases} \frac{1}{a_n} & \text{if } a_n \neq 0 \\ 0 & \text{if } a_n = 0 \end{cases}$	$M(2(q+1)+p)$

where 2^{p_x} is the upper bound of x provided by the Archimedian property.

Representation of real numbers $x \in [-1, 1]$

Dyadic rationals:

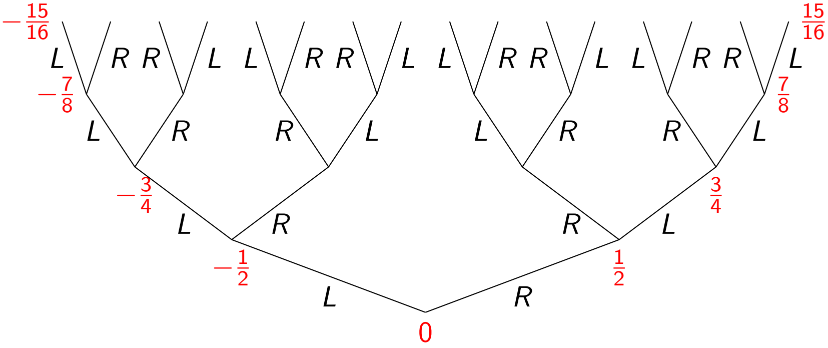
$$\sum_{n<m} \frac{k_n}{2^{n+1}} \qquad \text{with } k_n \in \{-1, 1\}.$$



with $\bar{1} := -1$. Adjacent dyadics can differ in many digits:

$$\frac{7}{16} \sim 1\bar{1}11, \qquad \frac{9}{16} \sim 11\bar{1}\bar{1}.$$

Cure: flip after 1. Binary reflected (or Gray-) code.



$$\frac{7}{16} \sim \mathrm{RRRL}, \qquad \frac{9}{16} \sim \mathrm{RLRL}.$$

Problem with productivity:

$$\bar{1}111 + 1\bar{1}\bar{1}\bar{1}\cdots = ? \qquad \text{(or } \mathrm{LRLL}\ldots + \mathrm{RRRL}\cdots = ?\text{)}$$

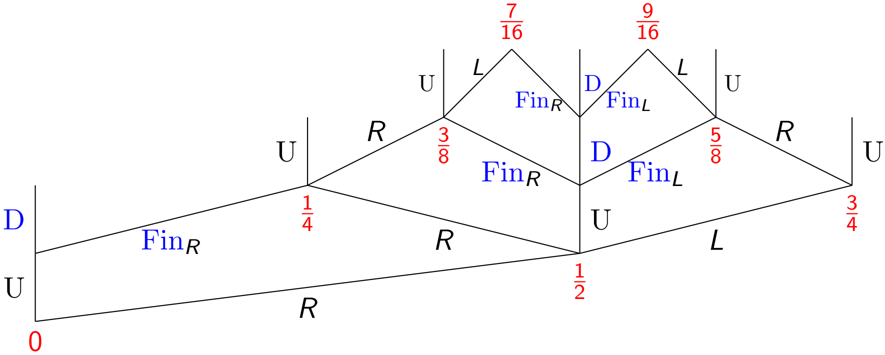
What is the first digit? Cure: delay.

- For binary code: add 0. Signed digit code

$$\sum_{n<m} \frac{k_n}{2^{n+1}} \qquad \text{with } k_n \in \{-1, 0, 1\}.$$

  Widely used for real number computation. There is a lot of redundancy: $\bar{1}1$ and $0\bar{1}$ both denote $-\frac{1}{4}$.
- For Gray-code: add U (undefined), D (delay), $\mathrm{Fin}_{L/R}$ (finally left / right). Pre-Gray code.

Pre-Gray code



Can remove Fin_a (by $\mathrm{U} \circ \mathrm{Fin}_a \mapsto a \circ \mathrm{R}$, $\mathrm{D} \circ \mathrm{Fin}_a \mapsto \mathrm{Fin}_a \circ \mathrm{L}$)

$$\mathrm{RRRLLL}\ldots \qquad \mathrm{RLRLLL}\ldots \qquad \mathrm{RUDDDD}\ldots$$

all denote $\frac{1}{2}$. Only keep the latter to denote $\frac{1}{2}$.
Result: unique representation, called pure Gray code.

Average for signed digit streams

Goal:

$$x, y \in {}^{\mathrm{co}}I \to \frac{x+y}{2} \in {}^{\mathrm{co}}I.$$

- Need to accomodate streams in our logical framework.
- Model streams as "cototal objects" in the (free) algebra **I** given by the single constructor $\mathrm{C}\colon \mathbf{SD} \to \mathbf{I} \to \mathbf{I}$.

Intuitively, $k_0, k_1, k_2 \dots$ represents

$$\sum_{n=0}^{\infty} \frac{k_n}{2^{n+1}} \qquad \text{with } k_n \in \{-1, 0, 1\}.$$

$$\Phi(X) := \{\, x \mid \exists_{k \in \mathrm{SD}} \exists_{x' \in X} (x = \frac{x' + k}{2}) \,\}.$$

Then

$$\begin{aligned} I &:= \mu_X \Phi(X) && \text{least fixed point} \\ {}^{\mathrm{co}}I &:= \nu_X \Phi(X) && \text{greatest fixed point} \end{aligned}$$

satisfy the (strengthened) axioms

$$\begin{aligned} \Phi(I \cap X) \subseteq X &\to I \subseteq X && \text{induction} \\ X \subseteq \Phi({}^{\mathrm{co}}I \cup X) &\to X \subseteq {}^{\mathrm{co}}I && \text{coinduction} \end{aligned}$$

("strengthened" because their hypotheses are weaker than the fixed point property $\Phi(X) = X$).

Goal: compute the average of two stream-coded reals. Prove

$$x, y \in {}^{\mathrm{co}}I \to \frac{x+y}{2} \in {}^{\mathrm{co}}I.$$

Computational content of this proof will be the desired algorithm.

Informal proof (from Ulrich Berger & Monika Seisenberger 2006). Define sets P, Q of averages, Q with a "carry" $i \in \mathbb{Z}$:

$$P := \{\, \frac{x+y}{2} \mid x, y \in {}^{\mathrm{co}}I \,\}, \quad Q := \{\, \frac{x+y+i}{4} \mid x, y \in {}^{\mathrm{co}}I, i \in \mathrm{SD}_2 \,\},$$

Suffices: Q satisfies the clause coinductively defining ${}^{\mathrm{co}}I$. Then by the greatest-fixed-point axiom for ${}^{\mathrm{co}}I$ we have $Q \subseteq {}^{\mathrm{co}}I$. Since also $P \subseteq Q$ we obtain $P \subseteq {}^{\mathrm{co}}I$, which is our claim.

Q satisfies the ${}^{\mathrm{co}}I$-clause:

$$i \in \mathrm{SD}_2 \to x, y \in {}^{\mathrm{co}}I \to \exists_{j \in \mathrm{SD}_2} \exists_{k \in \mathrm{SD}} \exists_{x', y' \in {}^{\mathrm{co}}I} \Big(\frac{x + y + i}{4} = \frac{\frac{x' + y' + j}{4} + k}{2} \Big).$$

Proof. Define $J, K \colon \mathbb{Z} \to \mathbb{Z}$ such that

$$i = J(i) + 4K(i), \qquad |J(i)| \le 2, \qquad |i| \le 6 \to |K(i)| \le 1.$$

Then we can relate $\frac{x+k}{2}$ and $\frac{x+y+i}{4}$ by

$$\frac{\frac{x+k}{2} + \frac{y+l}{2} + i}{4} = \frac{\frac{x+y+J(k+l+2i)}{4} + K(k+l+2i)}{2}.$$

By coinduction we obtain $Q \subseteq {}^{\mathrm{co}}I$:

$$\exists_{i \in \mathrm{SD}_2} \exists_{x,y \in {}^{\mathrm{co}}I} (z = \frac{x+y+i}{4}) \to z \in {}^{\mathrm{co}}I.$$

This gives our claim

$$x, y \in {}^{\mathrm{co}}I \to \frac{x+y}{2} \in {}^{\mathrm{co}}I.$$

Implicit algorithm. $P \subseteq Q$ computes the first "carry" $i \in \mathrm{SD}_2$ and the tails of the inputs. Then $f \colon \mathbf{SD}_2 \times \mathbf{I} \times \mathbf{I} \to \mathbf{I}$ defined corecursively by

$$f(i, \mathrm{C}_d(u), \mathrm{C}_e(v)) = \mathrm{C}_{K(k+l+2i)}(f(J(k+l+2i), u, v))$$

is called repeatedly and computes the average step by step. (Here $(k,d), (l,e) \in \mathrm{SD}^{\mathbf{r}}$).

Realizability

Define the realizability extension $\Phi^{\mathbf{r}}$ of Φ by

$$\Phi^{\mathbf{r}}(Y) := \{\, (x, u) \mid \exists_{(k,d)\in \mathrm{SD}^{\mathbf{r}}} \exists_{(x',u')\in Y} (x = \frac{x' + k}{2} \wedge u = \mathrm{C}_d(u')) \,\}$$

Let

$$\begin{aligned} I^{\mathbf{r}} &:= \mu_Y \Phi^{\mathbf{r}}(Y) \qquad && \text{least fixed point} \\ ({}^{\mathrm{co}}I)^{\mathbf{r}} &:= \nu_Y \Phi^{\mathbf{r}}(Y) \qquad && \text{greatest fixed point.} \end{aligned}$$

They satisfy the (strengthened) axioms

$$\begin{aligned} \Phi^{\mathbf{r}}(I^{\mathbf{r}} \cap Y) \subseteq Y \to I^{\mathbf{r}} \subseteq Y \qquad && \text{induction} \\ Y \subseteq \Phi^{\mathbf{r}}(({}^{\mathrm{co}}I)^{\mathbf{r}} \cup Y) \to Y \subseteq ({}^{\mathrm{co}}I)^{\mathbf{r}} \qquad && \text{coinduction.} \end{aligned}$$

From the proof M of

$$x, y \in {}^{\mathrm{co}}I \to \frac{x+y}{2} \in {}^{\mathrm{co}}I$$

extract a term $\mathrm{et}(M)$. The Soundness theorem gives a proof of

$$\mathrm{et}(M) \;\mathbf{r}\; \forall_{x,y}(x, y \in {}^{\mathrm{co}}I \to \frac{x+y}{2} \in {}^{\mathrm{co}}I).$$

Brouwer-Heyting-Kolmogorov interpretation:

$$u \;\mathbf{r}\; (x \in {}^{\mathrm{co}}I) \to v \;\mathbf{r}\; (y \in {}^{\mathrm{co}}I) \to \mathrm{et}(M)(u, v) \;\mathbf{r}\; (\frac{x+y}{2} \in {}^{\mathrm{co}}I).$$

This is a formal verification that $\mathrm{et}(M)$ computes the average w.r.t. signed digit streams.

Average for pre-Gray code

Method essentially the same as for signed digit streams.

- Only need to insert a different computational content to the predicates expressing how a real x is given.
- Instead of ${}^{\mathrm{co}}I$ for signed digit streams we now need two such predicates ${}^{\mathrm{co}}G$ and ${}^{\mathrm{co}}H$, corresponding to the two "modes" in pre-Gray code.

Method also works for multiplication and division:

$$
\begin{aligned}
&x, y \in {}^{\mathrm{co}}I \to \frac{x+y}{2} \in {}^{\mathrm{co}}I,\\
&x, y \in {}^{\mathrm{co}}I \to x \cdot y \in {}^{\mathrm{co}}I,\\
&x, y \in {}^{\mathrm{co}}I \to \frac{1}{4} \leq y \to \frac{x}{y} \in {}^{\mathrm{co}}I,
\end{aligned}
$$

both w.r.t. signed digit and Gray code.

Conclusion

- Want formally verified algorithms on real numbers given as streams (signed digits or pre-Gray code).
- Consider formal proofs M and apply realizability to extract their computational content.
- Switch between different representations of reals by relativising x to a coinductive predicate whose computational content is a stream representing x.
- The desired algorithm is obtained as the extracted term $\mathrm{et}(M)$ of the proof M.
- Verification by (automatically generated) formal soundness proof of the realizability interpretation.

References

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