Simultaneous inductive/coinductive definition of continuous functions

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Main points

- Semantics for constructive analysis based on the Scott/Ershov partial continuous functionals.
- Simultaneous inductive/coinductive definition of (uniformly) continuous functions.
- Extraction of computational content from proofs in an abstract theory.

Corecursion (1/2)

Example: transformation of an "abstract" real in $\mathbb{I} := [-1, 1]$ into a stream representation using "signed digits" $-1, 0, 1$.

- Assume an abstract (axiomatic) theory of reals, having an unspecified type ρ, and a type σ for rationals.
- Assume that the abstract theory provides us with a function $g\colon \rho \to \sigma \to \sigma \to \mathbf{B}$ comparing a real x with a proper rational interval $p < q$:

$$g(x, p, q) = \mathtt{tt} \to x \leq q,$$
$$g(x, p, q) = \mathtt{ff} \to p \leq x.$$

Corecursion (2/2)

- From g define a function $h\colon \rho \to \mathbf{U} + \rho + \rho + \rho$ by

$$h(x) := \begin{cases} \mathrm{inl}(\mathrm{inl}(\mathrm{inr}(2x+1))) & \text{if } g(x, -\frac{1}{2}, 0) = \mathrm{tt} \\ \mathrm{inl}(\mathrm{inr}(2x)) & \text{if } g(x, -\frac{1}{2}, 0) = \mathrm{ff},\ g(x, 0, \frac{1}{2}) = \mathrm{tt} \\ \mathrm{inr}(2x-1) & \text{if } g(x, 0, \frac{1}{2}) = \mathrm{ff} \end{cases}$$

- h is definable by a closed term M_h in Gödel's T.

Then the desired function f transforming an abstract real x into a stream can be defined by

$$f(x) := {}^{\mathrm{co}}\mathcal{R}^{\rho}_{\mathbf{I}} x M_h.$$

Free algebras

- **N** with constructors 0, S.
- **I** with constructors $\mathbb{I}$ (for $[-1,1]$) and $\mathrm{C}_{-1}, \mathrm{C}_0, \mathrm{C}_1$ (for the left, middle, right part of the interval, of half its length). For example, $\mathrm{C}_{-1}\mathbb{I}$, $\mathrm{C}_0\mathbb{I}$ and $\mathrm{C}_1\mathbb{I}$ denote $[-1,0]$, $[-\frac{1}{2},\frac{1}{2}]$ and $[0,1]$.
- $(\mathbf{W}, \mathbf{R})$ with constructors

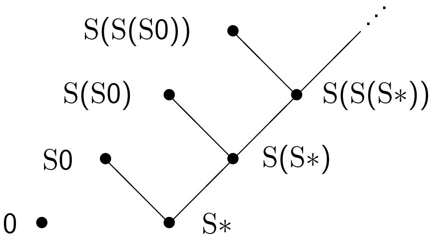
$W_0 \colon \mathbf{W}$	stop,
$W \colon \mathbf{R} \to \mathbf{W}$	quit writing and go into read mode,
$R_d \colon \mathbf{W} \to \mathbf{R}$	quit reading and write d ($d \in \{-1,0,1\}$),
$R \colon \mathbf{R} \to \mathbf{R} \to \mathbf{R} \to \mathbf{R}$	read the next digit and stay in read mode.

Information systems $\mathbf{C}_\rho$ for partial continuous functionals

- Types ρ: from algebras ι by $\rho \to \sigma$.
- $\mathbf{C}_\rho := (C_\rho, \mathrm{Con}_\rho, \vdash_\rho)$.
- Tokens a (= atomic pieces of information): constructor trees $\mathrm{C}a_1^*, \dots a_n^*$ with a_i^* a token or $*$. Example: $\mathrm{S}(\mathrm{S}*)$.
- Formal neighborhoods U: $\{a_1, \dots, a_n\}$, consistent.
- Entailment $U \vdash a$.

Ideals $x \in |\mathbf{C}_\rho|$ ("points", here: partial continuous functionals): consistent deductively closed sets of tokens.

Tokens and entailment for **N**



Constructors as continuous functions

- Every constructor C generates an ideal in the function space: $r_{\mathrm{C}} := \{ (U, \mathrm{C}a^*) \mid U \vdash a^* \}$. Associated continuous map:

$$|r_{\mathrm{C}}|(x) = \{ \mathrm{C}a^* \mid \exists_{U \subseteq x}(U \vdash a^*) \}.$$

- Constructors are injective and have disjoint ranges:

$$|r_{\mathrm{C}}|(\vec{x}\,) \subseteq |r_{\mathrm{C}}|(\vec{y}\,) \leftrightarrow \vec{x} \subseteq \vec{y},$$
$$|r_{\mathrm{C}_1}|(\vec{x}\,) \cap |r_{\mathrm{C}_2}|(\vec{y}\,) = \emptyset.$$

- Both properties are false for flat information systems (for them, by monotonicity, constructors need to be strict).

$$|r_{\mathrm{C}}|(\emptyset, y) = \emptyset = |r_{\mathrm{C}}|(x, \emptyset),$$
$$|r_{\mathrm{C}_1}|(\emptyset) = \emptyset = |r_{\mathrm{C}_2}|(\emptyset).$$

Total and cototal ideals of base type

- Total ideals of **I**:

$$\mathbb{I}_{\frac{i}{2^k},k} := [\frac{i}{2^k} - \frac{1}{2^k}, \frac{i}{2^k} + \frac{1}{2^k}] \qquad \text{for } -2^k < i < 2^k.$$

- Cototal ideals of **I**: reals in $[-1, 1]$, in (non-unique) stream representation using signed digits $-1, 0, 1$. Examples:

$$\begin{aligned} &\{\, \mathrm{C}_{-1}^n \mathbb{I} \mid n \geq 0 \,\} \quad \text{(representing the real } -1\text{)},\\ &\{\mathbb{I}\} \cup \{\, \mathrm{C}_1^n \mathrm{C}_{-1} \mathbb{I} \mid n \geq 0 \,\} \quad \text{(representing 0)},\\ &\{\mathbb{I}\} \cup \{\, \mathrm{C}_{-1}^n \mathrm{C}_1 \mathbb{I} \mid n \geq 0 \,\} \quad \text{(representing 0)}. \end{aligned}$$

- Cototal ideals x: every constructor tree $P(*) \in x$ has a "$\succ_1$-successor" $P(\mathrm{C}\vec{*}) \in x$.
- Total ideals: the cototal ones with $\succ_1$ well-founded.

$(\mathbf{W}, \mathbf{R})$ and continuous real functions

- Consider a well-founded "read tree", i.e., a constructor tree built from R (ternary) with R_d at its leaves. The digit d at a leaf means that, after reading all input digits on the path leading to the leaf, the output d is written.
- Let $R_{d_1}, \dots, R_{d_n}$ be all leaves. At a leaf R_{d_i} continue with W (i.e., write d_i), and continue reading.
- Result: a "**W**-cototal **R**-total" ideal; a representation of a uniformly continuous real function $f : \mathbb{I} \to \mathbb{I}$.
- Examples: $P := R(R_1 WP, R_0 WP, R_{-1} WP)$ represents the function $f(x) := -x$, and $R_0 WP$ represents the function $f(x) := -\frac{x}{2}$.

Formalization

- TCF: theory of computable functionals.
- Minimal logic ($\to$, $\forall$), plus inductive & coinductive definitions.
- Variables range over partial continuous functionals.
- Constants denote computable functionals.
- Terms: from T^+, a common of Gödel's T and Plotkin's PCF.

Cototality for $\mathbf{N}$

- In the algebra $\mathbf{N}$ define cototality coinductively by the clause

$$ {}^{\mathrm{co}}T_{\mathbf{N}}n \to \mathrm{Eq}(n,0) \vee \exists_m(\mathrm{Eq}(n,\mathrm{S}m) \wedge {}^{\mathrm{co}}T_{\mathbf{N}}m). $$

- Its greatest-fixed-point axiom (coinduction) is

$$ Pn \to \forall_n(Pn \to \mathrm{Eq}(n,0) \vee \exists_m(\mathrm{Eq}(n,\mathrm{S}m) \wedge Pm)) \to {}^{\mathrm{co}}T_{\mathbf{N}}n. $$

It expresses that every "competitor" P satisfying the same clause is a subset of ${}^{\mathrm{co}}T_{\mathbf{N}}$.

Cototality for $\mathbf{I}$

- In the algebra $\mathbf{I}$ of standard rational intervals cototality is defined by

$$
\begin{aligned}
{}^{\mathrm{co}}T_{\mathbf{I}}x \to \mathrm{Eq}(x, \mathbb{I}) \lor {} & \exists_y(\mathrm{Eq}(x, \mathrm{C}_{-1}y) \land {}^{\mathrm{co}}T_{\mathbf{I}}y) \lor \\
& \exists_y(\mathrm{Eq}(x, \mathrm{C}_0 y) \land {}^{\mathrm{co}}T_{\mathbf{I}}y) \lor \\
& \exists_y(\mathrm{Eq}(x, \mathrm{C}_1 y) \land {}^{\mathrm{co}}T_{\mathbf{I}}y).
\end{aligned}
$$

- A model is provided by the set of all finite or infinite streams of signed digits $-1, 0, 1$, i.e., the (non-unique) stream representation of real numbers.

A simultaneous inductive/coinductive definition (1/3)

- Example: formalization of an abstract theory of (uniformly) continuous real functions $f \colon \mathbb{I} \to \mathbb{I}$ where $\mathbb{I} := [-1, 1]$.
- Let Cf express that f is a continuous real function, and $\mathbb{I}_{p,k} := [p - 2^{-k}, p + 2^{-k}]$. Assume the abstract theory proves

$$Cf \to \forall_k \exists_l B_{l,k} f, \quad \text{with } B_{l,k} f := \forall_p \exists_q (f[\mathbb{I}_{p,l}] \subseteq \mathbb{I}_{q,k}).$$

- Let $\mathbb{I}_{-1} := [-1, 0]$, $\mathbb{I}_0 := [-\frac{1}{2}, \frac{1}{2}]$ and $\mathbb{I}_1 := [0, 1]$. Define in_d, out_d such that $\mathrm{in}_d[\mathbb{I}] = \mathbb{I}_d$ and $\mathrm{out}_d[\mathbb{I}_d] = \mathbb{I}$ by

$$\mathrm{in}_d(x) := \frac{d + x}{2}, \quad \mathrm{out}_d(x) := 2x - d.$$

Both functions are inverse to each other.

A simultaneous inductive/coinductive definition (2/3)

- Inductively define a predicate I_Y (Y a parameter) by

$$f[\mathbb{I}] \subseteq \mathbb{I}_d \to Y(\mathrm{out}_d \circ f) \to I_Y f \quad (d \in \{-1, 0, 1\}), \tag{1}$$

$$I_Y(f \circ \mathrm{in}_{-1}) \to I_Y(f \circ \mathrm{in}_0) \to I_Y(f \circ \mathrm{in}_1) \to I_Y f. \tag{2}$$

- The corresponding least-fixed-point axiom is

$$\begin{aligned}
&I_Y f \to \\
&(\forall_f(f[\mathbb{I}] \subseteq \mathbb{I}_d \to Y(\mathrm{out}_d \circ f) \to Pf))_{d\in\{-1,0,1\}} \to \\
&\forall_f((I_Y(f \circ \mathrm{in}_d))_{d\in\{-1,0,1\}} \to \\
&\qquad (P(f \circ \mathrm{in}_d))_{d\in\{-1,0,1\}} \to Pf) \to \\
&Pf).
\end{aligned} \tag{3}$$

A simultaneous inductive/coinductive definition (3/3)

- Using I_Y define a predicate J coinductively by

$$Jf \to \mathrm{Eq}(f, \mathrm{id}) \vee I_J f.$$

- The corresponding greatest-fixed-point axiom is

$$Qf \to \forall_f (Qf \to \mathrm{Eq}(f, \mathrm{id}) \vee I_Q f) \to Jf.$$

- Consider continuous functions $f \colon \mathbb{I} \to \mathbb{I}$, and let

$$B'_{l,k} f := \forall_p \exists_q (f[\mathbb{I}_{p,l} \cap \mathbb{I}] \subseteq \mathbb{I}_{q,k}).$$

- Lemma. (a). $B'_{l,k}(\mathrm{out}_d \circ f) \to B'_{l,k+1} f$.
  (b). Assume $B'_{l_d,k+1}(f \circ \mathrm{in}_d)$ for all $d \in \{-1, 0, 1\}$. Then $B'_{l,k+1}(f)$ with $l := 1 + \max_{d \in \{-1,0,1\}} l_d$.
- Proposition. (a). $\forall_f (Cf \to f[\mathbb{I}] \subseteq \mathbb{I} \to Jf)$.
  (b). $\forall_f (Jf \to f[\mathbb{I}] \subseteq \mathbb{I} \to \forall_k \exists_l B'_{l,k} f)$.

Realizability interpretation

- Realizability interpretation $t \mathbf{r} A$ by terms t in T^+.
- Soundness theorem.
- Decorations ($\to^{\mathrm{c}}, \forall^{\mathrm{c}}$ and $\to^{\mathrm{nc}}, \forall^{\mathrm{nc}}$) for fine-tuning:

$$
\begin{aligned}
t \mathbf{r} (A \to^{\mathrm{c}} B) &:= \forall_x (x \mathbf{r} A \to tx \mathbf{r} B),\\
t \mathbf{r} (A \to^{\mathrm{nc}} B) &:= \forall_x (x \mathbf{r} A \to t \mathbf{r} B),\\
t \mathbf{r} (\forall^{\mathrm{c}}_x A) &:= \forall_x (tx \mathbf{r} A),\\
t \mathbf{r} (\forall^{\mathrm{nc}}_x A) &:= \forall_x (t \mathbf{r} A).
\end{aligned}
$$

Decorating the existential quantifier

- $\exists_x A$ is inductively defined by the clause

$$\forall_x (A \to \exists_x A)$$

  with least-fixed-point axiom

$$\exists_x A \to \forall_x (A \to P) \to P.$$

- Decorations lead to variants $\exists^{\mathrm{d}}, \exists^{\mathrm{l}}, \exists^{\mathrm{r}}, \exists^{\mathrm{nc}}$ (d for "double", l for "left" and r for "right").

$$\begin{aligned}
&\forall^{\mathrm{c}}_x (A \to^{\mathrm{c}} \exists^{\mathrm{d}}_x A), && \exists^{\mathrm{d}}_x A \to^{\mathrm{c}} \forall^{\mathrm{c}}_x (A \to^{\mathrm{c}} P) \to^{\mathrm{c}} P,\\
&\forall^{\mathrm{c}}_x (A \to^{\mathrm{nc}} \exists^{\mathrm{l}}_x A), && \exists^{\mathrm{l}}_x A \to^{\mathrm{c}} \forall^{\mathrm{c}}_x (A \to^{\mathrm{nc}} P) \to^{\mathrm{c}} P,\\
&\forall^{\mathrm{nc}}_x (A \to^{\mathrm{c}} \exists^{\mathrm{r}}_x A), && \exists^{\mathrm{r}}_x A \to^{\mathrm{c}} \forall^{\mathrm{nc}}_x (A \to^{\mathrm{c}} P) \to^{\mathrm{c}} P,\\
&\forall^{\mathrm{nc}}_x (A \to^{\mathrm{nc}} \exists^{\mathrm{nc}}_x A), && \exists^{\mathrm{nc}}_x A \to^{\mathrm{nc}} \forall^{\mathrm{nc}}_x (A \to^{\mathrm{nc}} P) \to^{\mathrm{c}} P.
\end{aligned}$$

Computational content of coinductive definitions (1/3)

- Reconsider the example concerning “abstract” reals, having an unspecified type ρ.
- Assume the abstract theory proves that every real can be compared with a proper rational interval:

$$\forall^{\mathrm{c}}_{x \in R; p,q \in Q} (p < q \to x \leq q \vee p \leq x).$$

Computational content of coinductive definitions (2/3)

- Coinductively define a predicate J of arity (ρ) by the clause

$$\begin{aligned}\forall^{\mathrm{nc}}_x (Jx \to^{\mathrm{c}} \mathrm{Eq}(x,0) \lor {} & \exists^{\mathrm{r}}_y (\mathrm{Eq}(x, \frac{y-1}{2}) \land Jy) \lor \\ & \exists^{\mathrm{r}}_y (\mathrm{Eq}(x, \frac{y}{2}) \land Jy) \lor \\ & \exists^{\mathrm{r}}_y (\mathrm{Eq}(x, \frac{y+1}{2}) \land Jy)).\end{aligned}$$

- The greatest-fixed-point axiom for J is

$$\begin{aligned}\forall^{\mathrm{nc}}_x (Px \to^{\mathrm{c}} \forall^{\mathrm{nc}}_x (Px \to^{\mathrm{c}} \mathrm{Eq}(x,0) \lor {} & \exists^{\mathrm{r}}_y (\mathrm{Eq}(x, \frac{y-1}{2}) \land Py) \lor \\ & \exists^{\mathrm{r}}_y (\mathrm{Eq}(x, \frac{y}{2}) \land Py) \lor \\ & \exists^{\mathrm{r}}_y (\mathrm{Eq}(x, \frac{y+1}{2}) \land Py)) \to^{\mathrm{c}} Jx).\end{aligned}$$

Computational content of coinductive definitions (3/3)

- J's clause has the same form as the definition of cototality ${}^{\mathrm{co}}T_{\mathbf{I}}$ for $\mathbf{I}$; in particular, its "associated algebras" are the same.
- The types of the clause and the GFP axiom for J are

$$\iota \to \mathbf{U} + \iota + \iota + \iota, \qquad \tau \to (\tau \to \mathbf{U} + \tau + \tau + \tau) \to \iota,$$

  respectively, with ι the algebra associated with this clause (which is $\mathbf{I}$), and $\tau := \tau(Pr)$.
- The former is the type of the destructor for ι, and the latter is the type of the corecursion operator ${}^{\mathrm{co}}\mathcal{R}_{\iota}^{\tau}$.
- Proposition. (a) $\forall_x^{\mathrm{nc}}(Rx \to^{\mathrm{c}} Jx)$.
  (b). $\forall_x^{\mathrm{nc}}(Jx \to^{\mathrm{c}} \forall_k^{\mathrm{c}} B_k x)$ with $B_k x := \exists_q^{\mathrm{l}}(x \in \mathbb{I}_{q,k})$, i.e., x can be approximated by a rational q with accuracy 2^{-k}.

Decorating inductive/coinductive definitions (1/3)

- Decorate (1) – (3) as follows.

$$\forall^{\mathrm{nc}}_f (f[\mathbb{I}] \subseteq \mathbb{I}_d \to Y(\mathrm{out}_d \circ f) \to^{\mathrm{c}} I_Y f) \quad (d \in \{-1,0,1\}),$$

$$\forall^{\mathrm{nc}}_f (I_Y(f \circ \mathrm{in}_{-1}) \to^{\mathrm{c}} I_Y(f \circ \mathrm{in}_0) \to^{\mathrm{c}} I_Y(f \circ \mathrm{in}_1) \to^{\mathrm{c}} I_Y f).$$

$$\begin{aligned}
&\forall^{\mathrm{nc}}_f (I_Y f \to^{\mathrm{c}} \\
&\qquad (\forall^{\mathrm{nc}}_f (f[\mathbb{I}] \subseteq \mathbb{I}_d \to Y(\mathrm{out}_d \circ f) \to^{\mathrm{c}} Pf))_{d \in \{-1,0,1\}} \to^{\mathrm{c}} \\
&\qquad \forall^{\mathrm{nc}}_f ((I_Y(f \circ \mathrm{in}_d))_{d \in \{-1,0,1\}} \to^{\mathrm{c}} \\
&\qquad\qquad (P(f \circ \mathrm{in}_d))_{d \in \{-1,0,1\}} \to^{\mathrm{c}} Pf) \to^{\mathrm{c}} \\
&\qquad Pf).
\end{aligned}$$

- The types are, for $\alpha := \tau(Yf)$ and $\tau_P := \tau(Pr)$

$$\begin{aligned}
&\alpha \to \mathbf{R}(\alpha), \\
&\mathbf{R}(\alpha) \to \mathbf{R}(\alpha) \to \mathbf{R}(\alpha) \to \mathbf{R}(\alpha), \\
&\mathbf{R}(\alpha) \to (\alpha \to \tau_P)^3 \to (\mathbf{R}(\alpha)^3 \to \tau_P^3 \to \tau_P) \to \tau_P.
\end{aligned}$$

Decorating inductive/coinductive definitions (2/3)

- The simultaneous inductive/coinductive definition of J is decorated by
$$\forall^{\mathrm{nc}}_{f}(Jf \to^{\mathrm{c}} \mathrm{Eq}(f, \mathrm{id}) \vee I_J f)$$
and its greatest-fixed-point axiom by
$$\forall^{\mathrm{nc}}_{f}(Qf \to^{\mathrm{c}} \forall^{\mathrm{nc}}_{f}(Qf \to^{\mathrm{c}} \mathrm{Eq}(f, \mathrm{id}) \vee I_Q f) \to^{\mathrm{c}} Jf).$$
- The types are, for $\tau_Q := \tau(Qs)$
$$\begin{aligned} &\mathbf{W} \to \mathbf{U} + \mathbf{R}(\mathbf{W}), \\ &\tau_Q \to (\tau_Q \to \mathbf{U} + \mathbf{R}(\tau_Q)) \to \mathbf{W}. \end{aligned}$$

Decorating inductive/coinductive definitions (3/3)

- Substituting α by $\mathbf{W}$ and writing $\mathbf{R}$ for $\mathbf{R}(\mathbf{W})$ gives

$$
\begin{aligned}
&\mathbf{W} \to \mathbf{R},\\
&\mathbf{R} \to \mathbf{R} \to \mathbf{R} \to \mathbf{R},\\
&\mathbf{R} \to (\mathbf{W} \to \tau_P)^3 \to (\mathbf{R}^3 \to \tau_P^3 \to \tau_P) \to \tau_P,\\
&\mathbf{W} \to \mathbf{U} + \mathbf{R},\\
&\tau_Q \to (\tau_Q \to \mathbf{U} + \mathbf{R}(\tau_Q)) \to \mathbf{W}.
\end{aligned}
$$

- These are the types of the first three constructors for $\mathbf{R}$, the fourth constructor for $\mathbf{R}$, the recursion operator $\mathcal{R}_{\mathbf{R}}^{\tau_P}$, the destructor for $\mathbf{W}$ and the corecursion operator ${}^{\mathrm{co}}\mathcal{R}_{\mathbf{W}}^{\tau_Q}$.

Conclusion

TCF (theory of computable functionals) as a possible foundation for exact real arithmetic.

- Simply typed theory, with "lazy" free algebras as base types ($\Rightarrow$ constructors are injective and have disjoint ranges).
- Variables range over partial continuous functionals.
- Constants denote computable functionals (:= r.e. ideals).
- Minimal logic ($\to$, $\forall$), plus inductive & coinductive definitions.
- Computational content in abstract theories.
- Decorations ($\to^{\mathrm{c}}, \forall^{\mathrm{c}}$ and $\to^{\mathrm{nc}}, \forall^{\mathrm{nc}}$) for fine-tuning.

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