Completion of metric spaces

Seminar Konstruktive Analysis, Mathematisches Institut, LMU

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- We generalize our treatment of the reals to metric spaces.
- We consider metric spaces (X, d) with a distance function $d\colon X \to X \to \mathbb{R}$ satisfying reflexivity, symmetry and the triangle inequality.

- A Cauchy sequence with modulus need not converge to a point in the space.
- Therefore we define the completion of the given space, whose elements are all pairs $((u_n)_n, M)$ of a Cauchy sequence in X and a modulus $M\colon \mathbb{Z}^+ \to \mathbb{N}$.

- We extend the given distance function d to the completion and show that we obtain a metric space again.
- We show that this completion is complete, in the sense that now every Cauchy sequence with modulus has a limit.

Definition

A metric on a set X is a map $d\colon X \to X \to \mathbb{R}$ such that for all $u, u', u'' \in X$

(a) $d(u, u) = 0$ (reflexivity),

(b) $d(u, u') = d(u', u)$ (symmetry), and

(c) $d(u, u'') \leq d(u, u') + d(u', u'')$ (triangle inequality).

A metric space is a pair (X, d) consisting of a set X and a metric d on X. The real $d(u, u')$ is called distance of u and u' w.r.t. d.

Lemma (MetrUB, MetrLB)

Let (X, d) be a metric space. Then

$$|d(u, u'') - d(u', u'')| \leq d(u, u') \leq d(u, u'') + d(u', u'').$$

Proof.

From the triangle inequality and symmetry we obtain both

$$d(u, u'') - d(u', u'') \leq d(u, u'),$$
$$d(u', u'') - d(u, u'') \leq d(u, u')$$

and hence the first inequality. The second one follows immediately from the triangle inequality and symmetry. □

- The completion of a metric space (X, d) consists of all pairs $((u_n)_n, M)$ such that $(u_n)_n$ is a Cauchy sequence with modulus M, that is,

$$d(u_n, u_m) \leq \frac{1}{2^p} \quad \text{for } n, m \geq M(p).$$

- Let $\tilde{X}$ be the set of all such pairs, called points. We extend d to points $w = ((u_n)_n, M)$ and $w' = ((u'_n)_n, M')$ of $\tilde{X}$ by

$$\tilde{d}(w, w') := ((d(u_n, u'_n))_n, L).$$

  with $L(p) := \max(M(p+1), M'(p+1))$.
- The completion $(\tilde{X}, \tilde{d})$ of a metric space (X, d) is a metric space again.

- Goal: the completion $(\tilde{X}, \tilde{d})$ of a metric space (X, d) is complete.
- We explicitly define the limit of a modulated Cauchy sequence of points $w_n = ((u_{n,l})_l, N_n)$ in the completion.
- Assume that $(w_n)_n$ is a Cauchy sequence with monotone modulus M. Then

$$\mathrm{Lim}((w_n)_n, M) := ((u_{n,N_n(n)})_n, K)$$

with $K(p) := \max(M(p+1), p+2)$.

Lemma (L1, MCauchyConvMod)

Let (X, d) be a metric space and $(\tilde{X}, \tilde{d})$ its completion. Assume that $w = ((u_n)_n, M)$ is a point in $\tilde{X}$. Then

$$\tilde{d}(\iota_X(u_n), w) \leq \frac{1}{2^p} \quad \text{for } n \geq M(p).$$

Proof.

Fix n, p with $n \geq M(p)$. Let $x_m := d(u_n, u_m)$ and $x := \lim_m x_m$ with modulus $(M(q+1))_q$. By definition x is $\tilde{d}(\iota_X(u_n), w)$. The goal $x \leq \frac{1}{2^p}$ follows from the completeness of the real numbers. □

Lemma (L2, MCplSeqApprox)

Let (X, d) be a metric space and $(\tilde{X}, \tilde{d})$ its completion. Assume that $w_n = ((u_{n,l})_l, N_n)$ is a point in $\tilde{X}$ for all n. Let $u_n := u_{n,N_n(n)}$. Then

$$\tilde{d}(\iota_X(u_n), w_n) \leq \frac{1}{2^n} \quad \text{for all } n.$$

Proof.

This follows from L1. □

Lemma (L3, MCplCauchyApprox)

Let (X, d) be a metric space and $(\tilde{X}, \tilde{d})$ its completion. Let again $w_n := ((u_{n,l})_l, N_n)$ be points in $\tilde{X}$ for all n. Assume that $(w_n)_n$ is a Cauchy sequence in $(\tilde{X}, \tilde{d})$ with monotone modulus M. Let $u_n := u_{n,N_n(n)}$. Then

$$d(u_n, u_m) \leq \frac{1}{2^p}$$

for $n, m \geq \max(M(p+1), p+2)$.

Proof.

Recall that $d(u_n, u_m) = \tilde{d}(\iota_X(u_n), \iota_X(u_m))$ by Definition.

$$
\begin{aligned}
&\tilde{d}(\iota_X(u_n), \iota_X(u_m)) \\
&\leq \tilde{d}(\iota_X(u_n), w_n) + \tilde{d}(w_n, w_m) + \tilde{d}(w_m, \iota_X(u_m)) \\
&= \frac{1}{2^n} + \frac{1}{2^{p+1}} + \frac{1}{2^m} \quad \text{by L2} \\
&\leq \frac{1}{2^p} \quad \text{since } n, m \geq max(M(p+1), p+2).
\end{aligned}
$$

□

Lemma (L4, MCplLimMCpl)

Let (X, d) be a metric space and $(\tilde{X}, \tilde{d})$ its completion. Let $(w_n)_n$ be a Cauchy sequence in $\tilde{X}$ with monotone modulus M. Then its limit $\mathrm{Lim}((w_n)_n, M)$ is a point in $\tilde{X}$.

Proof.

It suffices to show that the explicitly defined limit is a modulated Cauchy sequence. But this has been done in L3. □

Lemma (L5, MCplCauchyConvMod)

Let (X, d) be a metric space and $(\tilde{X}, \tilde{d})$ its completion. Let $(w_n)_n$ be a Cauchy sequence in $\tilde{X}$ with monotone modulus M. Then $(w_n)_n$ converges to its limit $\mathrm{Lim}((w_n)_n, M)$.

Theorem (L6, MCplCompleteLim)

Let (X, d) be a metric space and $(\tilde{X}, \tilde{d})$ its completion. Let $(w_n)_n$ be a Cauchy sequence in $\tilde{X}$ with monotone modulus M. Then $(w_n)_n$ converges with the same modulus M to its limit $\mathrm{Lim}((w_n)_n, M)$ in $\tilde{X}$.

Proof.

This follows from L4 and L5. $\square$