Continuity in constructive analysis

Helmut Schwichtenberg

Mathematisches Institut, LMU, München

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Aim:

Constructive analysis, with constructions $\sim$ good algorithms.

Errett Bishop 1967: "Foundations of Constructive Analysis"

> *The modulus of continuity ω is an indispensable part of the definition of a continuous function on a compact interval, although sometimes it is not mentioned explicitly. In the same way, the moduli of continuity of the restrictions of f to each compact subinterval are indispensable parts of the definition of a continuous function f on a general interval.*

A continuous function $f\colon (X,\rho,Q) \to (Y,\sigma,R)$ for separable metric spaces is given by

$$h\colon Q \to \mathbb{N} \to R \qquad \text{approximating map}$$

plus $\alpha, \omega, \gamma, \delta$ depending on w, r (center and radius of a ball):

- $\alpha\colon Q \to \mathbb{Z}^+ \to \mathbb{Z}^+ \to \mathbb{N}$ such that $(h(u,n))_n$ (for $\rho(u,w) \le \frac{1}{2^r}$) is a Cauchy sequence with modulus $\alpha_{w,r}(p)$;
- a modulus $\omega\colon Q \to \mathbb{Z}^+ \to \mathbb{Z}^+ \to \mathbb{Z}^+$ of (uniform) continuity, such that for $n \ge \alpha_{w,r}(p)$ and $\rho(u,w), \rho(v,w) \le \frac{1}{2^r}$
$$\rho(u,v) \le \frac{2}{2^{\omega_{w,r}(p)}} \to \sigma(h(u,n), h(v,n)) \le \frac{1}{2^p};$$
- maps $\gamma\colon Q \to \mathbb{Z}^+ \to R$, $\delta\colon Q \to \mathbb{Z}^+ \to \mathbb{Z}^+$ such that $\gamma(w,r)$ and $\delta(w,r)$ are center and radius of a ball containing all $h(u,n)$ (for $\rho(u,w) \le \frac{1}{2^r}$):
$$\rho(u,w) \le \frac{1}{2^r} \to \sigma(h(u,n), \gamma(w,r)) \le \frac{1}{2^{\delta(w,r)}}.$$

$\alpha, \omega, \gamma, \delta$ are required to have monotonicity properties.

f given by type-1 data only.

Example: Inverse map $(0, \infty) \to \mathbb{R}$

Let $0 < c < d$, and q be minimal such that $\frac{1}{2^q} \leq c$. Then inv is given by

- the approximating map $h(a, n) := \frac{1}{a}$
- the Cauchy modulus $\alpha(c, d, p) := 0$
- the modulus $\omega(c, d, p) := p + 2q + 1$ of uniform continuity, for

  $$|a - b| \leq \frac{1}{2^{p+2q}} \to \left|\frac{1}{a} - \frac{1}{b}\right| = \left|\frac{b - a}{ab}\right| \leq \frac{1}{2^p},$$

  because $ab \geq \frac{1}{2^{2q}}$
- the center $\gamma(c, d) := \frac{c}{c^2 - d^2}$ and radius $\delta(c, d) := \frac{d}{c^2 - d^2}$ of a ball containing all $\frac{1}{a}$ for $|a - c| \leq d$.

- Application $f(x)$ must (and can) be defined separately, since the approximating map operates on approximations only.
- $f(x)$ is independent from w, r.
- Application is compatible with equality on real numbers:

$$x = y \to f(x) = f(y).$$

- f has ω as a modulus of uniform continuity:

$$|x - y| \leq \frac{1}{2^{\omega(p)}} \to |f(x) - f(y)| \leq \frac{1}{2^p}.$$

- Composition can be defined.

Algorithms in constructive proofs?

Theorem. Every totally bounded set $A \subseteq \mathbb{R}$ has an infimum y.

Proof.
Given $\varepsilon = \frac{1}{2^p}$, let $a_0 < a_1 < \cdots < a_{n-1}$ be an ε-net:
$\forall_{x \in A} \exists_{i<n} (|x - a_i| < \varepsilon)$. Let $b_p = \min\{\, a_i \mid i<n \,\}$. $y := \lim_p b_p$. $\square$

Corollary. $\inf_{x \in [a,b]} f(x)$ exists, for $f : [a, b] \to \mathbb{R}$ continuous.

Proof.
Given ε, pick $a = a_0 < a_1 < \cdots < a_{n-1} = b$ s.t. $a_{i+1} - a_i < \omega(\varepsilon)$.
Then $f(a_0), f(a_1), \ldots, f(a_{n-1})$ is an ε-net for f's range. $\square$

Many $f(a_i)$ need to be computed.

Aim: Get x with $f(x) = \inf_{y \in [a,b]} f(y)$ and a better algorithm, assuming convexity.

Intermediate value theorem

Let $a < b$ be rationals. If $f\colon [a, b] \to \mathbb{R}$ is continuous with $f(a) \leq 0 \leq f(b)$, and with a uniform modulus of increase

$$\frac{1}{2^p} < d - c \to \frac{1}{2^{p+q}} < f(d) - f(c),$$

then we can find $x \in [a, b]$ such that $f(x) = 0$.

Proof (trisection method).

1. Approximate Splitting Principle. Let x, y, z be given with $x < y$. Then $z \leq y$ or $x \leq z$.
2. IVTAux. Assume $a \leq c < d \leq b$, say $\frac{1}{2^p} < d - c$, and $f(c) \leq 0 \leq f(d)$. Construct c_1, d_1 with $d_1 - c_1 = \frac{2}{3}(d - c)$, such that $a \leq c \leq c_1 < d_1 \leq d \leq b$ and $f(c_1) \leq 0 \leq f(d_1)$.
3. IVTcds. Iterate the step $c, d \mapsto c_1, d_1$ in IVTAux.

Let $x = (c_n)_n$ and $y = (d_n)_n$ with the obvious modulus. As f is continuous, $f(x) = 0 = f(y)$ for the real number $x = y$. $\square$

Extracted term

```
[k0]
 left((cDC rat@@rat)(1@2)
      ([n1]
        (cId rat@@rat=>rat@@rat)
        ([cd3]
          [let cd4
            ((2#3)*left cd3+(1#3)*right cd3@
             (1#3)*left cd3+(2#3)*right cd3)
            [if (0<=(left cd4*left cd4-2+
                    (right cd4*right cd4-2))/2)
             (left cd3@right cd4)
             (left cd4@right cd3)]]))
      (IntToNat(2*k0)))
```

where cDC is a form of the recursion operator.

Kolmogorov 1932: "Zur Deutung der intuitionistischen Logik"

- View a formula A as a computational problem, of type $\tau(A)$, the type of a potential solution or "realizer" of A.
- Example: $\forall_n \exists_{m>n} \mathrm{Prime}(m)$ has type $\mathbf{N} \to \mathbf{N}$.

Express this view as invariance under relizability axioms

$$\mathrm{Inv}_A \colon A \leftrightarrow \exists_z (z \mathbf{\ r\ } A).$$

Consequences are choice and independence of premise (Troelstra):

$$\forall_x \exists_y A(y) \to \exists_f \forall_x A(fx) \qquad \text{for } A \text{ n.c.}$$
$$(A \to \exists_x B) \to \exists_x (A \to B) \qquad \text{for } A, B \text{ n.c.}$$

All these are realized by identities.

Derivatives

Let $f, g \colon I \to \mathbb{R}$ be continuous. g is called derivative of f with modulus $\delta_f \colon \mathbb{Z}^+ \to \mathbb{N}$ of differentiability if for $x, y \in I$ with $x < y$,

$$y \leq x + \frac{1}{2^{\delta_f(p)}} \to \left|f(y) - f(x) - g(x)(y - x)\right| \leq \frac{1}{2^p}(y - x).$$

A bound on the derivative of f serves as a Lipschitz constant of f:

Lemma (BoundSlope)

Let $f \colon I \to \mathbb{R}$ be continuous with derivative f'. Assume that f' is bounded by M on I. Then for $x, y \in I$ with $x < y$,

$$\left|f(y) - f(x)\right| \leq M(y - x).$$

Infimum of a convex function

Let $f, f' \colon [a, b] \to \mathbb{R}$ $(a < b)$ be continuous and f' derivative of f. Assume that f is strictly convex with witness q, in the sense that $f'(a) < 0 < f'(b)$ and

$$\frac{1}{2^p} < d - c \to \frac{1}{2^{p+q}} < f'(d) - f'(c).$$

Then we can find $x \in (a, b)$ such that $f(x) = \inf_{y \in [a,b]} f(y)$.

Proof.

- To obtain x, apply the intermediate value theorem to f'.
- To prove $\forall_{y \in [a,b]}(f(x) \leq f(y))$ (this is "non-computational", i.e., a Harrop formula) one can use the standard arguments in classical analysis (Rolle's theorem, mean value theorem). $\square$

Conclusion

Aim: constructive analysis, with constructions $\sim$ good algorithms. Then extract these algorithms from proofs (realizability).

- Use order locatedness: given $c < d$, for all u

$$\forall_{v \in V}(c \leq \rho(u, v)) \vee \exists_{v \in V}(\rho(u, v) \leq d).$$

- Avoid total boundedness (existence of ε-nets).

Generally

- View constructive analysis as an extension of classical analysis.
- Formalize proofs in TCF (based on the Scott-Ershov model of partial continuous functionals), extract algorithms (in Minlog).
- Data are important (real number, continuous function ...).
- Low type levels: continuous $f : \mathbb{R} \to \mathbb{R}$ determined by its values on the rationals $\mathbb{Q}$.