A theory of computable functionals

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Overview

- Formulas and predicates
- A theory of computable functionals
- Brouwer - Heyting - Kolmogorov and decorations
- The type of a formula or predicate
- Realizability
- Extracted terms

Simultaneously define formula forms and predicate forms

$$A, B ::= P\vec{r} \mid A \to B \mid \forall_x A,$$
$$P, Q ::= X \mid \{\, \vec{x} \mid A \,\} \mid \mu_X(\forall_{\vec{x}_i}((A_{i\nu})_{\nu<n_i} \to X\vec{r}_i))_{i<k}$$

Need restriction: X at most strictly positive in $A_{i\nu}$.

Strict positivity

We define Y occurs at most strictly positive in C, for C either a formula form or a predicate form.

$$\frac{\mathrm{SP}(Y,P)}{\mathrm{SP}(Y,P\vec{r})} \qquad \frac{Y \notin \mathrm{FPV}(A) \quad \mathrm{SP}(Y,B)}{\mathrm{SP}(Y,A \to B)} \qquad \frac{\mathrm{SP}(Y,A)}{\mathrm{SP}(Y,\forall_x A)}$$

For $C = X$ or $C = \{\, \vec{x} \mid A \,\}$

$$\mathrm{SP}(Y,X) \qquad \frac{\mathrm{SP}(Y,A)}{\mathrm{SP}(Y,\{\, \vec{x} \mid A \,\})}$$

For C an inductive predicate

$$\frac{\mathrm{SP}(Y,A_{i\nu}) \text{ for all } i<k,\ \nu<n_i}{\mathrm{SP}(Y,\mu_X(\forall_{\vec{x}_i}((A_{i\nu})_{\nu<n_i} \to X\vec{r}_i))_{i<k})}$$

Simultaneously define formulas and predicates

$$\frac{\mathrm{Pred}(P)}{\mathrm{F}(P\vec{r})} \qquad \frac{\mathrm{F}(A) \quad \mathrm{F}(B)}{\mathrm{F}(A \to B)} \qquad \frac{\mathrm{F}(A)}{\mathrm{F}(\forall_x A)}$$

For predicate variables or comprehension terms

$$\mathrm{Pred}(X) \qquad \frac{\mathrm{F}(A)}{\mathrm{Pred}(\{\, \vec{x} \mid A \,\})}$$

For inductive predicates

$$\frac{\mathrm{F}(A_{i\nu}) \text{ and } \mathrm{SP}(X, A_{i\nu}) \text{ for all } i<k,\ \nu<n_i}{\mathrm{Pred}(\mu_X(\forall_{\vec{x}_i}((A_{i\nu})_{\nu<n_i} \to X\vec{r}_i))_{i<k})}$$

where to avoid empty inductive predicates we also require

$$X \notin \mathrm{FPV}(A_{0\nu}) \text{ for all } \nu<n_0.$$

Let $\forall_{\vec{x}}((A_\nu(X))_{\nu<n} \to X\vec{r})$ be the *i*-th component of I. Call

$$I_i^+ : \forall_{\vec{x}}((A_\nu(I))_{\nu<n} \to I\vec{r})$$

the *i*-th clause (or introduction axiom) of I.

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Theory of computable functionals

TCF is the system in minimal logic for $\to$ and $\forall$, whose formulas are those in F above, and whose axioms are, for each I,

- all I_i^+
- $I^-\colon \forall_{\vec{x}}(I\vec{x} \to (\forall_{\vec{x}_i}((A_{i\nu}(I \cap X))_{\nu<n_i} \to X\vec{r}_i))_{i<k} \to X\vec{x}\,)$

where $I \cap X := \{\, \vec{x} \mid I\vec{x} \wedge X\vec{x} \,\}$ with $\wedge$ defined inductively below.

Equalities

(i) Defined function constants D are introduced by computation rules, written $l = r$, but intended as left-to-right rewrites.

(ii) Leibniz equality Eq inductively defined below.

(iii) Pointwise equality between partial continuous functionals can be defined inductively as well.

(iv) If l and r have a finitary algebra as their type, $l = r$ can be read as a boolean term, where $=$ is the decidable equality defined as a boolean-valued binary function.

In TCF formulas $A(r)$ and $A(s)$ are identified if $r, s \in \mathrm{T}^+$ have a common reduct.

Leibniz equality Eq

$$\mathrm{Eq}^+ : \forall_x \mathrm{Eq}(x^\rho, x^\rho)$$
$$\mathrm{Eq}^- : \forall_{x,y}(\mathrm{Eq}(x,y) \to \forall_x Xxx \to Xxy).$$

Compatibility of Eq: $\forall_{x,y}(\mathrm{Eq}(x,y) \to A(x) \to A(y))$.

Proof.

Use Eq^- with $\{\, x, y \mid A(x) \to A(y) \,\}$ for X. □

Define falsity by $\mathbf{F} := \mathrm{Eq}(\mathrm{ff}, \mathrm{tt})$.

Ex-falso-quodlibet: $\mathrm{TCF} \vdash \mathbf{F} \to A$ for $\mathrm{FPV}(A) = \emptyset$.

Proof.

1. Show $\mathbf{F} \to \mathrm{Eq}(x^\rho, y^\rho)$.

$\mathrm{Eq}(\mathcal{R}^\rho_{\mathbf{B}}\mathrm{ff}xy, \mathcal{R}^\rho_{\mathbf{B}}\mathrm{ff}xy)$	by Eq^+
$\mathrm{Eq}(\mathcal{R}^\rho_{\mathbf{B}}\mathrm{tt}xy, \mathcal{R}^\rho_{\mathbf{B}}\mathrm{ff}xy)$	by compatibility from $\mathrm{Eq}(\mathrm{ff}, \mathrm{tt})$
$\mathrm{Eq}(x^\rho, y^\rho)$	by conversion.

2. Show $\mathbf{F} \to A$, by induction on A. *Case* $I\vec{s}$.
Let K_0 be the nullary clause, with final conclusion $I\vec{t}$.
By IH from $\mathbf{F}$ we can derive all parameter premises, hence $I\vec{t}$.
From $\mathbf{F}$ we also have $\mathrm{Eq}(s_i, t_i)$ by 1.
Hence $I\vec{s}$ by compatibility.
The cases $A \to B$ and $\forall_x A$ are obvious. □

Lifting a boolean term $r^{\mathbf{B}}$ to a formula

Define

$$\mathrm{atom}(r^{\mathbf{B}}) := \mathrm{Eq}(r^{\mathbf{B}}, \mathtt{tt}).$$

This simplifies equational reasoning.

Example: by the computation rules the boolean term $\mathrm{S}r =_{\mathbf{N}} \mathrm{S}s$, i.e. $=_{\mathbf{N}}(\mathrm{S}r, \mathrm{S}s)$, is identified with $r =_{\mathbf{N}} s$. Hence: no need to prove

$$\mathrm{S}r =_{\mathbf{N}} \mathrm{S}s \to r =_{\mathbf{N}} s.$$

Existence

$\exists_x A$ can be inductively defined (Martin-Löf):

$$Ex(Y) := \mu_X(\forall_x(Yx^\rho \to X)).$$

Abbreviate $\mathrm{Ex}(\{ x^\rho \mid A \})$ by $\exists_x A$. Then

$$\begin{aligned} \exists^+ &: \quad \forall_x(Yx \to \exists_x Yx), \\ \exists^- &: \quad \exists_x Yx \to \forall_x(Yx \to X) \to X. \end{aligned}$$

Conjunction, disjunction

$$\begin{aligned}\mathrm{And}(Y,Z) &:= \mu_X(Y \to Z \to X),\\ \mathrm{Or}(Y,Z) &:= \mu_X(Y \to X,\ Z \to X).\end{aligned}$$

Abbreviate $\mathrm{And}(\{\mid A\},\{\mid B\})$ by $A \wedge B$ and $\mathrm{Or}(\{\mid A\},\{\mid B\})$ by $A \vee B$. Then

$$\begin{aligned}
\wedge^+ &: \quad Y \to Z \to Y \wedge Z,\\
\wedge^- &: \quad Y \wedge Z \to (Y \to Z \to X) \to X,\\
\vee^+_{0,1} &: \quad Y \to Y \vee Z, \qquad Z \to Y \vee Z,\\
\vee^- &: \quad Y \vee Z \to (Y \to X) \to (Z \to X) \to X.
\end{aligned}$$

Even numbers. Introduction axioms:

$$\mathrm{Even}(0), \qquad \forall_n(\mathrm{Even}(n) \to \mathrm{Even}(\mathrm{S}(\mathrm{S}n)))$$

Elimination axiom:

$$\forall_n(\mathrm{Even}(n) \to X0 \to \forall_n(\mathrm{Even}(n) \to Xn \to X(\mathrm{S}(\mathrm{S}n))) \to Xn).$$

Totality. Introduction axioms:

$$T_{\mathbf{N}}0, \qquad \forall_n(T_{\mathbf{N}}n \to T_{\mathbf{N}}(\mathrm{S}n)).$$

Elimination axiom:

$$\forall_n(T_{\mathbf{N}}n \to X0 \to \forall_n(T_{\mathbf{N}}n \to Xn \to X(\mathrm{S}n)) \to Xn).$$

- Every "competitor" X satisfying the clauses contains $T_{\mathbf{N}}$.
- Induction for $\mathbf{N}$, which only holds for total numbers.
- Fits the logical elimination rules (main part comes first).
- "Strengthened" step formula $\forall_n(T_{\mathbf{N}}n \to Xn \to X(\mathrm{S}n))$.

Transitive closure $\mathrm{TC}_{\prec}$

Let $\prec$ be a binary predicate variable. Introduction axioms:

$$\forall_{x,y}(x \prec y \to \mathrm{TC}_{\prec}(x,y)),$$
$$\forall_{x,y,z}(x \prec y \to \mathrm{TC}_{\prec}(y,z) \to \mathrm{TC}_{\prec}(x,z)).$$

Elimination axiom:

$$\forall_{x,y}(\mathrm{TC}_{\prec}(x,y) \to \forall_{x,y}(x \prec y \to Xxy) \to \forall_{x,y,z}(x \prec y \to \mathrm{TC}_{\prec}(y,z) \to Xyz \to Xxz) \to Xxy).$$

Relation of TCF to type theory

- Main difference: partial functionals are first class citizens.
- "Logic enriched": Formulas and types kept separate.
- Minimal logic: $\to, \forall$ only. $\mathrm{Eq}(x, y)$ (Leibniz equality), $\exists$, $\vee$, $\wedge$ inductively defined (Martin-Löf).
- $\mathbf{F} := \mathrm{Eq}(\mathrm{ff}, \mathrm{tt})$. Ex-falso-quodlibet: $\mathbf{F} \to A$ provable.
- "Decorations" $\to^{\mathrm{nc}}$, $\forall^{\mathrm{nc}}$ (i) allow abstract theory (ii) remove unused data.

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Brouwer - Heyting - Kolmogorov

Have $\to^{\pm}$, $\forall^{\pm}$, $I^{\pm}$. BHK-interpretation:

- p proves $A \to B$ if and only if p is a construction transforming any proof q of A into a proof $p(q)$ of B.
- p proves $\forall_{x^\rho} A(x)$ if and only if p is a construction such that for all a^ρ, $p(a)$ proves $A(a)$.

Leaves open:

- What is a "construction"?
- What is a proof of a prime formula?

Proposal:

- Construction: computable functional.
- Proof of a prime formula $I\vec{r}$: generation tree.

Example: generation tree for $\mathrm{Even}(6)$ should consist of a single branch with nodes $\mathrm{Even}(0)$, $\mathrm{Even}(2)$, $\mathrm{Even}(4)$ and $\mathrm{Even}(6)$.

Decoration

Which of the variables $\vec{x}$ and assumptions $\vec{A}$ are actually used in the "solution" provided by a proof of

$$\forall_{\vec{x}}(\vec{A} \to I\vec{r}\,)?$$

To express this we split each of $\to, \forall$ into two variants:

- a "computational" one $\to^{\mathrm{c}}, \forall^{\mathrm{c}}$ and
- a "non-computational" one $\to^{\mathrm{nc}}, \forall^{\mathrm{nc}}$ (with restricted rules)

and consider

$$\forall^{\mathrm{nc}}_{\vec{x}}\forall^{\mathrm{c}}_{\vec{y}}(\vec{A} \to^{\mathrm{nc}} \vec{B} \to^{\mathrm{c}} X\vec{r}\,).$$

This will lead to a different (simplified) algebra ι_I associated with the inductive predicate I.

Examples

Write $\to$ if it does not matter whether we have $\to^{\mathrm{c}}$ or $\to^{\mathrm{nc}}$.

$$
\begin{aligned}
&\mathrm{Eq}^{+}: && \forall^{\mathrm{nc}}_{x}\mathrm{Eq}(x^{\rho}, x^{\rho})\\
&\mathrm{Eq}^{-}: && \forall^{\mathrm{nc}}_{x,y}(\mathrm{Eq}(x,y) \to \forall^{\mathrm{nc}}_{x} Xxx \to^{\mathrm{c}} Xxy),\\
&(\exists^{\mathrm{u}})^{+}: && \forall^{\mathrm{nc}}_{x}(Yx \to^{\mathrm{nc}} \exists^{\mathrm{u}}_{x} Yx),\\
&(\exists^{\mathrm{u}})^{-}: && \exists^{\mathrm{u}}_{x} Yx \to \forall^{\mathrm{nc}}_{x}(Yx \to^{\mathrm{nc}} X) \to^{\mathrm{c}} X,\\
&(\wedge^{\mathrm{u}})^{+}: && Y \to^{\mathrm{nc}} Z \to^{\mathrm{nc}} Y \wedge^{\mathrm{u}} Z,\\
&(\wedge^{\mathrm{u}})^{-}: && Y \wedge^{\mathrm{u}} Z \to (Y \to^{\mathrm{nc}} Z \to^{\mathrm{nc}} X) \to^{\mathrm{c}} X.
\end{aligned}
$$

Computational variants of existence and conjunction

$\exists^{\mathrm{u}}$ and $\wedge^{\mathrm{u}}$ have just been defined.

$$
\begin{array}{ll}
\forall^{\mathrm{c}}_x(Yx \to^{\mathrm{c}} \exists^{\mathrm{d}}_x Yx), & \exists^{\mathrm{d}}_x Yx \to^{\mathrm{c}} \forall^{\mathrm{c}}_x(Yx \to^{\mathrm{c}} X) \to^{\mathrm{c}} X, \\
\forall^{\mathrm{c}}_x(Yx \to^{\mathrm{nc}} \exists^{\mathrm{l}}_x Yx), & \exists^{\mathrm{l}}_x Yx \to^{\mathrm{c}} \forall^{\mathrm{c}}_x(Yx \to^{\mathrm{nc}} X) \to^{\mathrm{c}} X, \\
\forall^{\mathrm{nc}}_x(Yx \to^{\mathrm{c}} \exists^{\mathrm{r}}_x Yx), & \exists^{\mathrm{r}}_x Yx \to^{\mathrm{c}} \forall^{\mathrm{nc}}_x(Yx \to^{\mathrm{c}} X) \to^{\mathrm{c}} X,
\end{array}
$$

and similar for $\wedge$:

$$
\begin{array}{ll}
Y \to^{\mathrm{c}} Z \to^{\mathrm{c}} Y \wedge^{\mathrm{d}} Z, & Y \wedge^{\mathrm{d}} Z \to^{\mathrm{c}} (Y \to^{\mathrm{c}} Z \to^{\mathrm{c}} X) \to^{\mathrm{c}} X, \\
Y \to^{\mathrm{c}} Z \to^{\mathrm{nc}} Y \wedge^{\mathrm{l}} Z, & Y \wedge^{\mathrm{l}} Z \to^{\mathrm{c}} (Y \to^{\mathrm{c}} Z \to^{\mathrm{nc}} X) \to^{\mathrm{c}} X, \\
Y \to^{\mathrm{nc}} Z \to^{\mathrm{c}} Y \wedge^{\mathrm{r}} Z, & Y \wedge^{\mathrm{r}} Z \to^{\mathrm{c}} (Y \to^{\mathrm{nc}} Z \to^{\mathrm{c}} X) \to^{\mathrm{c}} X
\end{array}
$$

Computational variants of disjunction

$$
\begin{aligned}
&Y \to^{\mathrm{c}} Y \vee^{\mathrm{d}} Z, && Z \to^{\mathrm{c}} Y \vee^{\mathrm{d}} Z,\\
&Y \to^{\mathrm{c}} Y \vee^{\mathrm{l}} Z, && Z \to^{\mathrm{nc}} Y \vee^{\mathrm{l}} Z,\\
&Y \to^{\mathrm{nc}} Y \vee^{\mathrm{r}} Z, && Z \to^{\mathrm{c}} Y \vee^{\mathrm{r}} Z,\\
&Y \to^{\mathrm{nc}} Y \vee^{\mathrm{u}} Z, && Z \to^{\mathrm{nc}} Y \vee^{\mathrm{u}} Z
\end{aligned}
$$

with elimination axioms

$$
\begin{aligned}
&Y \vee^{\mathrm{d}} Z \to^{\mathrm{c}} (Y \to^{\mathrm{c}} X) \to^{\mathrm{c}} (Z \to^{\mathrm{c}} X) \to^{\mathrm{c}} X,\\
&Y \vee^{\mathrm{l}} Z \to^{\mathrm{c}} (Y \to^{\mathrm{c}} X) \to^{\mathrm{c}} (Z \to^{\mathrm{nc}} X) \to^{\mathrm{c}} X,\\
&Y \vee^{\mathrm{r}} Z \to^{\mathrm{c}} (Y \to^{\mathrm{nc}} X) \to^{\mathrm{c}} (Z \to^{\mathrm{c}} X) \to^{\mathrm{c}} X,\\
&Y \vee^{\mathrm{u}} Z \to^{\mathrm{c}} (Y \to^{\mathrm{nc}} X) \to^{\mathrm{c}} (Z \to^{\mathrm{nc}} X) \to^{\mathrm{c}} X.
\end{aligned}
$$

Each inductive predicate is marked as computationally relevant (c.r.) or non-computational (n.c.) (or Harrop). In the latter case:

- it is "uniform one-clause defined", i.e., has just one clause with $\forall^{\mathrm{nc}}, \to^{\mathrm{nc}}$ only (examples: Eq, $\exists^{\mathrm{u}}$, $\wedge^{\mathrm{u}}$), or
- it is a "witnessing predicate" $\boldsymbol{I}^{\mathbf{r}}$, or
- all clauses are "non-computational invariant" (no $\exists, \vee$).

Notation in the final case: $\mu^{\mathrm{nc}}_X(K_0, \dots, K_{k-1})$. Elimination scheme must be restricted to n.c. formulas.

Examples of n.c. inductive predicates are Eq, $\exists^{\mathrm{u}}$, $\wedge^{\mathrm{u}}$, $\vee^{\mathrm{nc}}$ where

$$(\vee^{\mathrm{nc}})^+_0 \colon Y \to^{\mathrm{nc}} Y \vee^{\mathrm{nc}} Z, \qquad (\vee^{\mathrm{nc}})^+_1 \colon Z \to^{\mathrm{nc}} Y \vee^{\mathrm{nc}} Z.$$

Note that $\vee^{\mathrm{u}}$ is c.r.

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The type $\tau(C)$ of a formula or predicate C, and ι_I

$\tau(C)$ type or the "nulltype symbol" $\circ$. Extend use of $\rho \to \sigma$ to $\circ$:

$$(\rho \to \circ) := \circ, \quad (\circ \to \sigma) := \sigma, \quad (\circ \to \circ) := \circ.$$

Assume a global assignment of a type variable ξ to every X.

$$\begin{aligned}
&\tau(P\vec{r}) := \tau(P),\\
&\tau(A \to^{\mathrm{c}} B) := (\tau(A) \to \tau(B)), \quad \tau(A \to^{\mathrm{nc}} B) := \tau(B),\\
&\tau(\forall^{\mathrm{c}}_{x^\rho} A) := (\rho \to \tau(A)), \quad \tau(\forall^{\mathrm{nc}}_{x^\rho} A) := \tau(A),\\
&\tau(X) := \xi,\\
&\tau(\{\, \vec{x} \mid A \,\}) := \tau(A),\\
&\tau(\mu^{\mathrm{nc}}_X(K_0, \dots, K_{k-1})) := \circ,\\
&\tau(\underbrace{\mu_X(\forall^{\mathrm{nc}}_{\vec{x}_i}\forall^{\mathrm{c}}_{\vec{y}_i}(\vec{A}_i \to^{\mathrm{nc}} \vec{B}_i \to^{\mathrm{c}} X\vec{r}_i))_{i<k}}_{I}) := \underbrace{\mu_\xi(\tau(\vec{y}_i) \to \tau(\vec{B}_i) \to \xi)_{i<k}}_{\iota_I}.
\end{aligned}$$

Call ι_I the algebra associated with I.

Examples

Let $a, b \in \mathbf{Q}$, $x \in \mathbf{R}$, $k \in \mathbf{Z}$, $f \in \mathbf{R} \to \mathbf{R}$.

- The formula

$$\forall^{\mathrm{c}}_{a,b,x}(a < b \to x \leq b \vee^{\mathrm{u}} a \leq x)$$

  has type $\mathbf{Q} \to \mathbf{Q} \to \mathbf{R} \to \mathbf{B}$.

- The formula

$$\begin{aligned} &\forall^{\mathrm{c}}_{f,k}(f(0) \leq 0 \leq f(1) \to \\ &\qquad \forall_{a,b}(\frac{1}{2^k}|b - a| \leq |f(b) - f(a)|) \to \\ &\qquad \exists^{\mathrm{l}}_x f(x){=}0) \end{aligned}$$

  has type $(\mathbf{R} \to \mathbf{R}) \to \mathbf{Z} \to \mathbf{R}$.

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Realizability

Introduce a special nullterm symbol ε to be used as a "realizer" for n.c. formulas. Extend term application to ε by

$$\varepsilon t := \varepsilon, \quad t\varepsilon := t, \quad \varepsilon\varepsilon := \varepsilon.$$

Assume a global assignment giving for every predicate variable X of arity $\vec{\rho}$ a predicate variable $X^{\mathbf{r}}$ of arity $(\tau(X), \vec{\rho})$.

$$\begin{aligned}
&t \mathbf{\ r\ } X\vec{r} := X^{\mathbf{r}} t\vec{r}, \\
&t \mathbf{\ r\ } (A \to^{\mathrm{c}} B) := \forall_x (x \mathbf{\ r\ } A \ \to\ tx \mathbf{\ r\ } B), \\
&t \mathbf{\ r\ } (A \to^{\mathrm{nc}} B) := \forall_x (x \mathbf{\ r\ } A \ \to\ t \mathbf{\ r\ } B), \\
&t \mathbf{\ r\ } \forall^{\mathrm{c}}_x A := \forall_x (tx \mathbf{\ r\ } A), \\
&t \mathbf{\ r\ } \forall^{\mathrm{nc}}_x A := \forall_x (t \mathbf{\ r\ } A), \\
&t \mathbf{\ r\ } \underbrace{(\mu_X(K_0, \dots, K_{k-1}))}_{I} \vec{s} := I^{\mathbf{r}} t\vec{s}
\end{aligned}$$

In case A is n.c., $\forall_x (x \mathbf{\ r\ } A \to B(x))$ means $\varepsilon \mathbf{\ r\ } A \to B(\varepsilon)$.

For

$$I := \mu_X(\forall^{\mathrm{nc}}_{\vec{x}_i}\forall^{\mathrm{c}}_{\vec{y}_i}((A_{i\nu})_{\nu<n_i} \to^{\mathrm{nc}} (B_{i\nu})_{\nu<m_i} \to^{\mathrm{c}} X\vec{r}_i))_{i<k}$$

define

$$I^{\mathbf{r}} := \{\, w, \vec{x} \mid (\mu^{\mathrm{nc}}_X(\forall^{\mathrm{nc}}_{\vec{x}_i,\vec{y}_i,\vec{u}_i}((\exists_{u_{i\nu}} u_{i\nu} \mathbf{\ r\ } A_{i\nu})_{\nu<n_i} \to^{\mathrm{nc}} (v_{i\nu} \mathbf{\ r\ } B_{i\nu})_{\nu<m_i} \to X(\mathrm{C}_i\vec{y}_i\vec{v}_i)\vec{r}_i))_{i<k})w\vec{x}\,\}.$$

For a general n.c. inductive predicate (with restricted elimination scheme) we define $\varepsilon \mathbf{\ r\ } I\vec{s}$ to be $I\vec{s}$. For the special n.c. inductive predicates $I^{\mathbf{r}}$, Eq, $\exists^{\mathrm{u}}$ and $\wedge^{\mathrm{u}}$ let

$$\begin{aligned}
\varepsilon \mathbf{\ r\ } I^{\mathbf{r}}t\vec{s} &:= I^{\mathbf{r}}t\vec{s},\\
\varepsilon \mathbf{\ r\ } \mathrm{Eq}(t,s) &:= \mathrm{Eq}(t,s),\\
\varepsilon \mathbf{\ r\ } \exists^{\mathrm{u}}_x A &:= \exists^{\mathrm{u}}_{x,y}(y \mathbf{\ r\ } A),\\
\varepsilon \mathbf{\ r\ } (A \wedge^{\mathrm{u}} B) &:= \exists^{\mathrm{u}}_x(x \mathbf{\ r\ } A) \wedge^{\mathrm{u}} \exists^{\mathrm{u}}_y(y \mathbf{\ r\ } B).
\end{aligned}$$

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For a derivation M of a formula A we define its extracted term $\mathrm{et}(M)$, of type $\tau(A)$. For M^A with A n.c. let $\mathrm{et}(M^A) := \varepsilon$. Else

$$
\begin{aligned}
\mathrm{et}(u^A) &:= x_u^{\tau(A)} \quad (x_u^{\tau(A)} \text{ uniquely associated to } u^A),\\
\mathrm{et}((\lambda_{u^A} M^B)^{A \to^{\mathrm{c}} B}) &:= \lambda_{x_u}^{\tau(A)} \mathrm{et}(M),\\
\mathrm{et}((M^{A \to^{\mathrm{c}} B} N^A)^B) &:= \mathrm{et}(M)\mathrm{et}(N),\\
\mathrm{et}((\lambda_{x^\rho} M^A)^{\forall_x^{\mathrm{c}} A}) &:= \lambda_x^\rho \mathrm{et}(M),\\
\mathrm{et}((M^{\forall_x^{\mathrm{c}} A(x)} r)^{A(r)}) &:= \mathrm{et}(M) r,\\
\mathrm{et}((\lambda_{u^A} M^B)^{A \to^{\mathrm{nc}} B}) &:= \mathrm{et}(M),\\
\mathrm{et}((M^{A \to^{\mathrm{nc}} B} N^A)^B) &:= \mathrm{et}(M),\\
\mathrm{et}((\lambda_{x^\rho} M^A)^{\forall_x^{\mathrm{nc}} A}) &:= \mathrm{et}(M),\\
\mathrm{et}((M^{\forall_x^{\mathrm{nc}} A(x)} r)^{A(r)}) &:= \mathrm{et}(M).
\end{aligned}
$$

Here $\lambda_{x_u}^{\tau(A)} \mathrm{et}(M)$ means $\mathrm{et}(M)$ if A is n.c.

Extracted terms for the axioms.

- Let I be c.r.

$$\mathrm{et}(I_i^+) := \mathrm{C}_i, \qquad \mathrm{et}(I^-) := \mathcal{R},$$

  where both C_i and $\mathcal{R}$ refer to the algebra ι_I associated with I.
- Let I be a general n.c. predicate. Take ε for both the clauses and the (restricted!) elimination axiom.
- For the witnessing predicate $I^{\mathbf{r}}$ define $\mathrm{et}((I^{\mathbf{r}})^-) := \mathcal{R}_{\iota_I}$.
- For Eq, $\exists^{\mathrm{u}}$, $\wedge^{\mathrm{u}}$ take identities of the appropriate type.

Theorem (Soundness)

*Let M be a derivation of A from assumptions $u_i : C_i$. Then we can derive $\mathrm{et}(M)$ **r** A from assumptions x_{u_i} **r** C_i.*

Proof.

By induction on M. □

- The derivation in TCF of $\mathrm{et}(M)$ **r** A can be machine checked (automated verification).
- Coq's extraction returns Ocaml code.
- Agda views (complete) proofs as programs.