Minimal from classical proofs

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Roadmap

Goal: an uncommon but useful approach to logic: minimal logic +

$\tilde{\exists}, \tilde{\vee}$ (weak, classical) and $\exists, \vee$ (strong, constructive).

1. Embedding classical and intuitionistic logic into minimal logic.
2. Geometric formulas G and geometric implications Γ.
   $\Gamma \vdash_c G$ implies $\Gamma \vdash_i G$.
3. Extended geometric implications: $\to$ occurs positively only.
4. $\tilde{\exists}, \tilde{\vee}$ versus $\exists, \vee$: variants of Barr's theorem.
5. Examples.

Minimal logic

- Minimal logic $\sim$ simply typed λ-calculus.
- Rules $\to^+$, $\to^-$, $\forall^+$, $\forall^-$.
- $\exists, \vee, \wedge$ inductively defined. Equivalent: defined by rules.
- Semantic: Beth structures. Correct and complete.

$\nvdash ((P \to Q) \to P) \to P$	Peirce
$\nvdash (P \to \exists_x Qx) \to \exists_x (P \to Qx)$	Independence of premise

derivation	term
$u\colon A$	u^A
$$\dfrac{\begin{matrix}[u\colon A] \\ \mid M \\ B\end{matrix}}{A \to B}\to^+ u$$	$(\lambda_{u^A} M^B)^{A\to B}$
$$\dfrac{\begin{matrix}\mid M \\ A \to B\end{matrix} \qquad \begin{matrix}\mid N \\ A\end{matrix}}{B}\to^-$$	$(M^{A\to B} N^A)^B$

Natural deduction: $\forall$-rules

derivation	term
$\dfrac{\begin{array}{c} \mid M \\ A \end{array}}{\forall_x A}\,\forall^+ x$ (Variable Cond.)	$(\lambda_x M^A)^{\forall_x A}$ (Variable Cond.)
$\dfrac{\begin{array}{c} \mid M \\ \forall_x A(x) \end{array} \qquad r}{A(r)}\,\forall^-$	$(M^{\forall_x A(x)} r)^{A(r)}$

Natural deduction: $\vee$, $\exists$-rules

$$\dfrac{\begin{array}{c} \mid M \\ A \end{array}}{A \vee B}\,\vee_0^+ \qquad \dfrac{\begin{array}{c} \mid M \\ B \end{array}}{A \vee B}\,\vee_1^+ \qquad \dfrac{\begin{array}{ccc} & [u\colon A] & [v\colon B] \\ \mid M & \mid N & \mid K \\ A \vee B & C & C \end{array}}{C}\,\vee^- u, v$$

$$\dfrac{\begin{array}{cc} & \mid M \\ r & A(r) \end{array}}{\exists_x A(x)}\,\exists^+ \qquad \dfrac{\begin{array}{cc} & [u\colon A] \\ \mid M & \mid N \\ \exists_x A & B \end{array}}{B}\,\exists^- x, u \text{ (var.cond.)}$$

Dragalin-Friedman-translation

Fix a formula A. Define B^A by

- $P^A := P \vee A$ for prime formulas P;
- $(B \circ C)^A := B^A \circ C^A$ for $\circ = \to, \wedge, \vee$;
- $(\forall_x B)^A := \forall_x B^A$ and $(\exists_x B)^A := \exists_x B^A$.

Easy:

$$\Gamma \vdash B \text{ implies } \Gamma^A \vdash B^A.$$

Embedding classical and intuitionistic logic

Fix $\bot$, and define $\neg A := A \to \bot$.

$$\begin{array}{lll} \mathrm{Stab}_P\colon \forall_{\vec{x}}(\neg\neg P\vec{x} \to P\vec{x}\,) & \text{stability,} \\ \mathrm{Efq}_P\colon \forall_{\vec{x}}(\bot \to P\vec{x}\,) & \text{ex-falso-quodlibet.} \end{array}$$

Let $\mathrm{Stab} := \{\,\mathrm{Stab}_P \mid P \text{ relation}\,\}$, $\mathrm{Efq} := \{\,\mathrm{Efq}_P \mid P \text{ relation}\,\}$.
Easy: Stab proves $\neg\neg A \to A$ for A built with $\to, \forall$ only.

- Define $\Gamma \vdash_c A$ by $\Gamma \cup \mathrm{Stab} \vdash A$.
- Define $\Gamma \vdash_i A$ by $\Gamma \cup \mathrm{Efq} \vdash A$.

Define

$$\tilde{\exists}_x A := \neg\forall_x\neg A, \qquad A \mathbin{\tilde{\vee}} B := \neg(A \to B \to \bot).$$

Negative translation by Gödel-Gentzen

A^g is defined by

- $\bot^g := \bot$;
- $P^g := \neg\neg P$ for prime formulas $P \neq \bot$ (where $\neg A := A \to \bot$);
- $(B \vee C)^g := B^g \mathbin{\tilde{\vee}} C^g$;
- $(\exists_x B)^g := \tilde{\exists}_x B^g$;
- $(B \circ C)^g := B^g \circ C^g$ for $\circ = \to, \wedge$;
- $(\forall_x B)^g := \forall_x B^g$.

Easy:

$$\Gamma \vdash_c B \text{ implies } \Gamma^g \vdash B^g.$$

Geometric implications

Geometric formulas are defined by

$$G, H ::= P \mid \bot \mid G \wedge H \mid G \vee H \mid \exists_x G.$$

A geometric implication has the form $\forall_{\vec{x}}(G \to H)$.

- Every geometric formula can be written in the form

$$\exists_{\vec{x}}(B_1 \vee \cdots \vee B_n)$$

  with B_i conjunctions of prime formulas.
- Every geometric implication can be written as conjunction of

$$\forall_{\vec{x}}(B \to \exists_{\vec{y}}(B_1 \vee \cdots \vee B_n))$$

  with B, B_i conjunctions of prime formulas.

Theorem. (Palmgren 2002). Let Γ be geometric implications, G a geometric formula. $\Gamma \vdash_c G$ implies $\Gamma \vdash \bot \vee G$, hence $\Gamma \vdash_i G$.

$$G^g \leftrightarrow \neg\neg G \tag{1}$$

$$(G \to H) \to G^g \to H^g \tag{2}$$

$$G^A \leftrightarrow G \vee A \tag{3}$$

$$(G \to H) \to G^A \to H^A \tag{4}$$

$$G^G \leftrightarrow G. \tag{5}$$

$\Gamma^g \vdash G^g$	Gödel-Gentzen
$\Gamma \vdash G^g$	by (2)
$\Gamma \vdash \neg\neg G$	by (1)
$\Gamma^G \vdash (\neg\neg G)^G$	Dragalin-Friedman, $A := G$
$\Gamma \vdash (\neg\neg G)^G$	by (4)
$(\neg\neg G)^G \leftrightarrow ((G^G \to \bot \vee G) \to \bot \vee G)$	for $\neg\neg G = (G \to \bot) \to \bot$
$G^G \to G$	by (5)

Corollary. Let Γ be geometric implications, G a geometric formula. If $\Gamma \vdash_c G$ and Γ, G have no $\bot$, then $\Gamma \vdash G$.

Proof.

- Suppose Γ, G have no $\bot$.
- Have proof of G from Γ and instances $\bot \to A_i$ of Efq.
- Replace $\bot$ by $\bigwedge\!\!\!\bigwedge_i A_i$.
- This does not affect Γ, G and turns each $\bot \to A_i$ into a provable formula.

$\square$

Extended geometric implications (EGI)

are formulas containing $\to$ only positively. Every geometric implication is an EGI. Examples of non-EGIs:

- A double negation $(A \to \bot) \to \bot$, or
- the premise of the Peirce formula $((P \to Q) \to P) \to P$.

Theorem

For EGIs Γ and A, classical derivability of A from Γ implies intuitionistic derivability.

Theorem

For EGIs Γ, A without disjunction, classical derivability of A from Γ implies derivability in minimal logic.

Let $A_1, \dots, A_n$ be EGIs. We transform a classical proof in long normal form of a prime formula Q from $A_1, \dots, A_n$ into an intuitionistic proof of Q from the same assumptions.

Lemma

Let $A_1, \dots, A_n$ be EGIs. Consider a proof in long normal form of an implication-free formula, from the assumptions $A_1, \dots, A_n$ and stability axioms. Let u be an assumption variable bound by $\to^+$. Then u is bound in a context $\mathrm{Stab}_P \vec{r} S(\lambda_u M^\perp)$, where $u \colon \neg P\vec{r}$ and $\lambda_u M^\perp$ is a top node of the segment S.

Proof. Consider the path in the proof whose initial node is u. The path must have an order > 0, and the binding position of u must be in the introduction part of a path of possibly smaller order, ending in the side premise of an $\to^-$-rule:

$$\dfrac{\begin{matrix} \\ \mid N \\ A_1 \to B_1 \end{matrix} \qquad \begin{matrix} u\colon A \\ \mid M \\ A_1 \end{matrix}}{B_1} \to^-$$

Therefore an implication $A \to B$ is a (strictly) positive subformula of A_1 and hence a negative subformula of $A_1 \to B_1$. By our assumptions this is only possible if N is a stability axiom applied to some terms $\vec{r}$. Hence we have the situation

$$\dfrac{\dfrac{\mathrm{Stab}_P\colon \forall_{\vec{x}}(\neg\neg P\vec{x} \to P\vec{x}\,) \qquad \vec{r}}{\neg\neg P\vec{r} \to P\vec{r}} \qquad \begin{matrix} \mid M \\ \dfrac{\bot}{\neg\neg P\vec{r}} \to^+ u \\ \mid S \\ \neg\neg P\vec{r} \end{matrix}}{P\vec{r}} \to^-$$

with $u\colon \neg P\vec{r}$ and $\lambda_u M^\perp$ a top node of the segment S.

Barr's theorem

- Recall: our language has $\exists, \vee$ and $\tilde{\exists}, \tilde{\vee}$.
- Let $\tilde{A}$ be obtained from A by replacing $\exists, \vee$ with $\tilde{\exists}, \tilde{\vee}$.

Lemma

$\vdash_c \tilde{A} \leftrightarrow A^g$.

Theorem (Barr)

Assume that Γ consists of geometric implications and G is a geometric formula. $\tilde{\Gamma} \vdash_c \tilde{G}$ implies $\Gamma \vdash \bot \vee G$, hence also $\Gamma \vdash_i G$. If Γ, G do not contain $\bot$, we have $\Gamma \vdash G$.

Proof (Palmgren).

By the lemma $\tilde{\Gamma} \vdash_c \tilde{G}$ is equivalent to $\Gamma^g \vdash_c G^g$. Hence $\Gamma^g \vdash G^g$ by the Gödel-Gentzen translation, since $\vdash (A^g)^g \leftrightarrow A^g$. But Palmgren's proof above began with $\Gamma^g \vdash G^g$. $\square$

Variants of Barr's theorem

$$\vdash \exists_x A \to \tilde{\exists}_x A, \qquad \vdash A \vee B \to A \mathbin{\tilde{\vee}} B,$$

but not conversely; this is the reason why $\tilde{\exists}, \tilde{\vee}$ are called "weak".
For formulas A possibly with both $\exists, \vee$ and $\tilde{\exists}, \tilde{\vee}$ we define strengthenings A^+ and weakenings A^-:

$$A^+ : \quad \text{replace in } A \quad \begin{cases} \text{some positive occurrences of } \tilde{\exists}, \tilde{\vee} & \text{by } \exists, \vee, \\ \text{some negative occurrences of } \exists, \vee & \text{by } \tilde{\exists}, \tilde{\vee}, \end{cases}$$

$$A^- : \quad \text{replace in } A \quad \begin{cases} \text{some positive occurrences of } \exists, \vee & \text{by } \tilde{\exists}, \tilde{\vee}, \\ \text{some negative occurrences of } \tilde{\exists}, \tilde{\vee} & \text{by } \exists, \vee. \end{cases}$$

Lemma

- $\vdash A^+ \to A$,
- $\vdash A \to A^-$.

One proves both parts simultaneously by induction on A.

Theorem

Assume $\tilde{\Gamma} \vdash_c P$ where Γ has only positive occurrences of $\to, \exists, \vee$ and P is a prime formula. Then $\Gamma \vdash_i P$.

Proof.

By the Lemma $\Gamma \vdash \tilde{\Gamma}$, so $\Gamma \vdash_c P$, so $\Gamma \vdash_i P$ (theorem above). $\square$

Theorem

Assume $\tilde{\Gamma} \vdash_c \tilde{A}$ where

(i) *$A = \exists_{\vec{x}}(B_1 \vee \dots \vee B_n)$ with $B_i = \forall_{\vec{y}_i} C_i$ and C_i conjunction of prime formulas $\neq \perp$;*

(ii) *Γ has only positive occurrences of $\to, \exists$ and contains neither $\vee$ nor $\perp$.*

Then $\Gamma \vdash A$.

Geometric formulas and geometric implications almost (no $\vee, \perp$) have the required form. We allow e.g. prenex formulas with prime formulas as kernel for Γ (needed in [Coquand & Lombardi 2006]), and universally quantified prime formulas for the disjunctive B_i's.

Proof.

Assume $\tilde{\Gamma} \vdash_c \tilde{A}$ with A, Γ satisfying (i), (ii). By the Lemma and (ii) we have $\Gamma \vdash \tilde{\Gamma}$, hence $\Gamma \vdash_c \tilde{A}$. By (i) we can assume that $\tilde{A}$ has the form $\tilde{\exists}_{\vec{x}_1} B_1 \tilde{\vee} \dots \tilde{\vee} \tilde{\exists}_{\vec{x}_n} B_n$ with $B_i = \forall_{\vec{y}_i} C_i$ and C_i conjunction of prime formulas $\neq \bot$. Therefore

$$\Gamma, \forall_{\vec{x}_1}(\forall_{\vec{y}_1} C_1 \to \bot), \dots, \forall_{\vec{x}_n}(\forall_{\vec{y}_n} C_n \to \bot) \vdash_c \bot.$$

By a theorem above

$$\Gamma, \forall_{\vec{x}_1}(\forall_{\vec{y}_1} C_1 \to \bot), \dots, \forall_{\vec{x}_n}(\forall_{\vec{y}_n} C_n \to \bot) \vdash \bot.$$

Replace $\bot$ by $\exists_{\vec{x}_1} \forall_{\vec{y}_1} C_1 \vee \dots \vee \exists_{\vec{x}_n} \forall_{\vec{y}_n} C_n$. This leaves Γ intact and makes the final premises provable. Hence $\Gamma \vdash A$ as required. □

Examples

- Warning: care is needed when interpreting the results.
- Reason: we have $\exists, \vee$ as well as $\tilde{\exists}, \tilde{\vee}$, and defined $\Gamma \vdash_c A$ to mean $\Gamma \cup \mathrm{Stab} \vdash A$.
- The common understanding of classical derivability of A from Γ is written here as $\tilde{\Gamma} \vdash_c \tilde{A}$, which is different from $\Gamma \vdash_c A$.
- $\vdash_c P \mathbin{\tilde{\vee}} \neg P$, but $\nvdash_c P \vee \neg P$.
- $\vdash_c (\forall_x Qx \to P) \to \tilde{\exists}_x (Qx \to P)$, but $\nvdash_c (\forall_x Qx \to P) \to \exists_x (Qx \to P)$, for otherwise this formula (an EGI) would be derivable in minimal logic, which it isn't.

Examples

- However, this caution is not necessary in the $\to, \forall$-fragment.
- For instance, $\forall_x(\forall_y Rxy \to Qx) \to P$ is derivable in classical logic iff it is derivable in minimal logic.
- Premises called "generalized Horn clauses", CSL 1991.
- Related work by Dale Miller: extensions of logic programming.

Examples in algebra

[Coquand & Lombardi 2006] consider a non-Noetherian version of Swan's theorem, written in a first-order way as an implication

$$\mathrm{Hdim} R < n \to \Delta_n(F) = 1 \to \exists_{X,Y}(1 = XFY) \tag{6}$$

(X a row vector, Y a column vector, F a matrix of fixed size). $\mathrm{Hdim} R < n$ (the *Heitmann dimension* of R is $< n$); for $n = 1$:

$$\forall_x \exists_a \forall_y \exists_b (1 = b(1 - yx(1 - ax))). \tag{7}$$

(n given) is an EGI, as is $\Delta_n(F) = 1$ and $\exists_{X,Y}(1 = XFY)$. Notice: In (6) and its premise (7) we have $\exists$, i.e., $\mathrm{Hdim} R < 1$ has a constructive meaning. But here the variant of Barr's theorem can be applied: (6) with $\exists$ replaced by $\tilde{\exists}$ is provable in minimal logic. (As reported in Mints' survey [1991], conservativity of classical over intuitionistic logic for such formulas follows from Orevkov [1968]).

Examples in algebra (continued)

[Coquand and Quitté 2011] consider a theorem of Hilbert-Burch: if we have an exact sequence

$$0 \to R^2 \to R^3 \to \langle a_1, a_2, a_3 \rangle \to 0,$$

then the elements a_1, a_2, a_3 have a gcd, i.e.,

$$\exists_x (x | a_1 \wedge x | a_2 \wedge x | a_3 \wedge \forall_y (y | a_1 \wedge y | a_2 \wedge y | a_3 \to y | x)).$$

Not a geometric formula. But: $x | y$ can be axiomatized as a prime formula. Consider rings with a prime formula $R(a_1, a_2, a_3, x, y)$ s.t.

$$R(a_1, a_2, a_3, x, y) \leftrightarrow (y | a_1 \wedge y | a_2 \wedge y | a_3 \to y | x).$$

Then the conclusion is $\exists_x \forall_y (x | a_1 \wedge x | a_2 \wedge x | a_3 \wedge R(a_1, a_2, a_3, x, y))$, and hence again the variant of Barr's theorem applies.

Necessity of the conditions

Clearly the (left) nesting of implications must be restricted, since

- $\vdash_c \neg\neg P \to P$ but $\nvdash_i \neg\neg P \to P$
- $\vdash_c ((P \to Q) \to P) \to P$ but $\nvdash ((P \to Q) \to P) \to P$.

We cannot allow implications in the conclusion. Kreisel's example:

- Let R be a primitive recursive such that $\exists_z Rxz$ is undecidable.
- $\vdash_c \tilde{\exists}_y \forall_z (Rxz \to Rxy)$ (this holds even for $\vdash$).
- But there is no computable f s.t. $\forall_{x,z}(Rxz \to R(x, f(x)))$, for then $\exists_z Rxz$ would be decidable: it is true iff $R(x, f(x))$ holds.
- Hence $\nvdash \exists_y \forall_z (Rxz \to Rxy)$.

Conclusion

- An uncommon approach to logic: minimal logic +

$$\tilde{\exists}, \tilde{\vee} \text{ (weak, classical) and } \exists, \vee \text{ (strong, constructive).}$$

- The view of classical logic as minimal logic + stability allows to prove new conservativity results (for EGIs).
- Corollaries: useful variants of Barr's theorem.

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