# A pointfree theory of partial continuous functionals

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# Finitary algebras as non-flat Scott information systems

- An algebra $\iota$ is given by its constructors.
- Examples:

$$0^{\mathbf{N}}, \mathrm{S}^{\mathbf{N} \to \mathbf{N}} \quad \text{for } \mathbf{N} \text{ (unary natural numbers)},$$
$$1^{\mathbf{P}}, S_0^{\mathbf{P} \to \mathbf{P}}, S_1^{\mathbf{P} \to \mathbf{P}} \quad \text{for } \mathbf{P} \text{ of (binary positive numbers)},$$
$$0^{\mathbf{D}} \text{ (axiom) and } \mathrm{C}^{\mathbf{D} \to \mathbf{D} \to \mathbf{D}} \text{ (rule) for } \mathbf{D} \text{ (derivations)}.$$

- Examples of "information tokens": $\mathrm{S}^n 0$ ($n \geq 0$), $\mathrm{S}^2 *$ (in **N**), $\mathrm{C}(\mathrm{C}0*)(\mathrm{C}*0)$ (in **D**) ($*$: special symbol; no information).
- An information token is total if it contains no $*$.
- In **D**: total token $\sim$ finite (well-founded) derivation.

# Finitary algebras: consistency, entailment, ideals

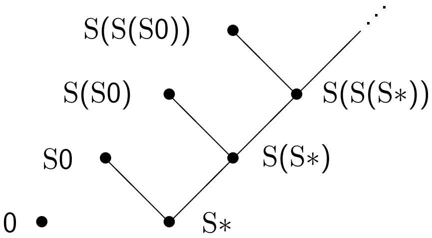
For **D** (derivations):

- $\{C0*, C*0\}$ is "consistent", written $C0* \uparrow C*0$.
- $\{C0*, C*0\} \vdash C00$ ("entails").
- Ideals: consistent and "deductively closed" sets of tokens.

Examples of ideals:

- $\{C0*, C**\}$.
- $\{C00, C0*, C*0, C**\}$.
- The deductive closure of a finite (well-founded) derivation.
- $\{C**, C(C**)*, C*(C**), C(C**)(C**), \dots\}$ ("cototal").
- Locally correct, but possibly non well-founded derivations (Mints 1978).

# Tokens and entailment for **N**



# Constructors as continuous functions

- Continuous maps $f\colon |\mathbf{N}| \to |\mathbf{N}|$ (see below) are monotone: $x \subseteq y \to fx \subseteq fy$.
- Easy: every constructor gives rise to a continuous function.
- Want: constructors have disjoint ranges and are injective (cf. the Peano axioms $\mathrm{S}x \neq 0$ and $\mathrm{S}x = \mathrm{S}y \to x = y$).
- This holds for non-flat algebras, but not for flat ones:


There constructors must be strict (i.e., $\mathrm{C}\vec{x}\emptyset\vec{y} = \emptyset$), hence

| | | |
|---|---|---|
| In **P**: | $\mathrm{S}_1\emptyset = \emptyset = \mathrm{S}_2\emptyset$ | (overlapping ranges), |
| In **D**: | $\mathrm{C}\emptyset\{0\} = \emptyset = \mathrm{C}\{0\}\emptyset$ | (not injective). |

# The Scott-Ershov model of partial continuous functionals

- Let $\mathbf{A} = (A, \mathrm{Con}_A, \vdash_A)$, $\mathbf{B} = (B, \mathrm{Con}_B, \vdash_B)$ be information systems (Scott). Function space: $\mathbf{A} \to \mathbf{B} := (C, \mathrm{Con}, \vdash)$, with

$$C := \mathrm{Con}_A \times B,$$

$$\{(U_i, b_i)\}_{i \in I} \in \mathrm{Con} := \forall_{J \subseteq I}(\bigcup_{j \in J} U_j \in \mathrm{Con}_A \to \{b_j\}_{j \in J} \in \mathrm{Con}_B),$$

$$\{(U_i, b_i)\}_{i \in I} \vdash (U, b) := (\{\, b_i \mid U \vdash_A U_i \,\} \vdash_B b).$$

- Partial continuous functionals of type $\rho$: the ideals in $\mathbf{C}_\rho$.

$$\mathbf{C}_\iota := (\mathrm{Tok}_\iota, \mathrm{Con}_\iota, \vdash_\iota), \qquad \mathbf{C}_{\rho \to \sigma} := \mathbf{C}_\rho \to \mathbf{C}_\sigma.$$

- $f \in |\mathbf{C}_\rho|$: limit of formal neighborhoods $U \in \mathrm{Con}_{\rho \to \sigma}$.
- $f \in |\mathbf{C}_\rho|$ computable: r.e. limit.

# Formal language, axioms

- Base theory: $\mathrm{HA}$ with prim. rec. functions, induction scheme.
- Code types $\rho$, tokens $a$, formal neighborhoods $U$.
- $U$ as $\{ a_i \mid i < n \}$, finite enumerated set ($a_i$ prim. rec.).
- $\Delta$-formula: equation $t = 0$ with $t$ prim. rec. term.
- Fix $W = \{ (U_i, b_i) \mid i < n \}$, $z := \{ a \mid C(a) \}$ ($C$ $\Delta$-formula).

$$Wz := \{ b_i \mid \forall_{a \in U_i} C(a) \}$$

can be written as a finite enumerated set.

# Formal language, axioms (continued)

- (Typed) variables $x, y, f, g$ for sets of tokens (points).
- (Typed) terms are built from variables and constants:

$$M, N ::= x^\rho \mid \mathrm{C}^\rho \mid (\lambda_{x^\rho} M^\sigma)^{\rho \to \sigma} \mid (M^{\rho \to \sigma} N^\rho)^\sigma.$$

- Σ-formula: $t = 0$ (Δ-formula), $a \in x$; $\wedge$, $\vee$, $\exists$, $\forall_{i<n}$.
- An ideal is a consistent deductively closed set of tokens.

$$\mathrm{Ideal}(x) := \forall_{a,b \in x}(a \uparrow b) \wedge \forall_{U \subseteq x} \forall_a (U \vdash a \to a \in x).$$

- Σ-comprehension. Let $C(a, \vec{y})$ be a Σ-formula.

$$\begin{aligned} \mathrm{Ideal}(\vec{y}) &\to \forall_{a,b}(C(a,\vec{y}) \to C(b,\vec{y}) \to a \uparrow b) \\ &\to \forall_{U,b}(\forall_{a \in U} C(a,\vec{y}) \to U \vdash b \to C(b,\vec{y})) \\ &\to \exists_x \forall_a (a \in x \leftrightarrow C(a,\vec{y})). \end{aligned}$$

# Definition of $a \in M$

For every closed term $\lambda_{\vec{x}} M$ of type $\vec{\rho} \to \sigma$ we define a $\Sigma$-formula $(\vec{U}, a) \in \lambda_{\vec{x}} M$ $\quad ((\vec{U}, a)$ denotes $(U_1, \dots (U_n, a) \dots))$.

$$\frac{U_i \vdash a}{(\vec{U}, a) \in \lambda_{\vec{x}} x_i} (V),$$

$$\frac{(\vec{U}, V, c) \in \lambda_{\vec{x}} M \qquad \forall_{b \in V}((\vec{U}, b) \subseteq \lambda_{\vec{x}} N)}{(\vec{U}, c) \in \lambda_{\vec{x}}(MN)} (A),$$

$$\frac{\vec{V} \vdash \vec{a^*}}{(\vec{U}, \vec{V}, \mathrm{C}\vec{a^*}) \in \lambda_{\vec{x}} \mathrm{C}} (\mathrm{C}) \quad \text{for every constructor } \mathrm{C}.$$

Then

$$(a \in M) := \exists_{\vec{U} \subseteq \vec{x}}((\vec{U}, a) \in \lambda_{\vec{x}} M).$$

This is independent of the choice of $\vec{x}$.

# Terms and ideals, extensional equality, totality

- Lemma. Every term $M$ denotes an ideal, i.e.,

$$\mathrm{Ideal}(\vec{x}) \to a, b \in M \to a \uparrow b,$$
$$\mathrm{Ideal}(\vec{x}) \to U \subseteq M \to U \vdash b \to b \in M.$$

- $(M = N) := \forall_a (a \in M \leftrightarrow a \in N)$.
- $G_\rho x$ ($x$ is a total ideal) is defined by induction on $\rho$:

$$G_\iota x := \mathrm{Ideal}(x) \land x \text{ contains a total token } a,$$
$$G_{\rho \to \sigma} f := \mathrm{Ideal}(f) \land \forall_x (G_\rho x \to \underbrace{\exists_y (y = fx \land G_\sigma y)}_{G_\sigma (fx)}).$$

# Properties

Extension lemma: If $f \in G_\rho$, $\mathrm{Ideal}(g)$ and $f \subseteq g$, then $g \in G_\rho$. (Proof. *Case* $\rho \to \sigma$. Let $f \in G_{\rho\to\sigma}$ and $f \subseteq g$. Show $g \in G_{\rho\to\sigma}$. Let $x \in G_\rho$. Show $gx \in G_\sigma$. But $gx \supseteq fx \in G_\sigma$. Use IH.)

Continuity of application: $b \in fx \leftrightarrow \exists_{U \subseteq x}((U, b) \in f)$.

$$\begin{aligned} b \in (f_1 \cap f_2)x &\leftrightarrow \exists_{U \subseteq x}((U, b) \in f_1 \cap f_2) \\ &\leftrightarrow_{(*)} \exists_{U_1 \subseteq x}((U_1, b) \in f_1) \land \exists_{U_2 \subseteq x}((U_2, b) \in f_2) \\ &\leftrightarrow b \in f_1 x \land b \in f_2 x. \\ &\leftrightarrow b \in (f_1 x) \cap (f_2 x). \end{aligned}$$

"$\to$" of $(*)$ is obvious. For "$\leftarrow$", let $U_i \subseteq x$ with $(U_i, b) \in f_i$. Choose $U = U_1 \cup U_2$. Then $(U, b) \in f_i$ (as $\{(U_i, b)\} \vdash (U, b)$ and $f_i$ is deductively closed).

# Equality $=^t_\rho$ on total ideals

$$(x =^t_\iota y) := (x = y), \quad (f =^t_{\rho\to\sigma} g) := \forall_{x \in G_\rho}(fx =^t_\sigma gx).$$

Lemma. $f =^t_\rho g$ iff $f \cap g \in G_\rho$, for $f, g \in G_\rho$. Proof. *Case* $\rho \to \sigma$.

$$\begin{aligned} f =^t_{\rho\to\sigma} g &\leftrightarrow \forall_{x \in G_\rho}(fx =^t_\sigma gx) \\ &\leftrightarrow \forall_{x \in G_\rho}(fx \cap gx \in G_\sigma) \quad \text{by IH} \\ &\leftrightarrow \forall_{x \in G_\rho}((f \cap g)x \in G_\sigma) \\ &\leftrightarrow f \cap g \in G_{\rho\to\sigma}. \end{aligned}$$

## Theorem (Ershov 1974, Longo & Moggi 1984)

*$x =^t_\rho y$ implies $fx =^t_\sigma fy$, for $x, y \in G_\rho$ and $f \in G_{\rho\to\sigma}$.*

Proof. Since $x =^t_\rho y$ we have $x \cap y \in G_\rho$ by the previous lemma. Now $fx, fy \supseteq f(x \cap y)$ and hence $fx \cap fy \in G_\sigma$. This implies $fx =^t_\sigma fy$ again by the previous lemma.

# Density

The total functionals are dense (w.r.t. the Scott topology) in the space of all partial continuous functionals of type $\rho$.

Theorem (Kreisel 1959, Ershov 1974, U. Berger 1993)

*For every type $\rho = \rho_1 \to \ldots \to \rho_p \to \iota$ we have $\Delta$-formulas $\mathrm{TExt}_\rho$ and $\mathrm{Sep}^i_\rho$ $(i = 1, \ldots, p)$ such that for any given $U, V \in \mathrm{Con}_\rho$*

(a) $U \subseteq \{\, a \mid \mathrm{TExt}_\rho(U, a) \,\} \in G_\rho$ *and*

(b) $U \not\uparrow_\rho V \to \vec{z}_{U,V} \in G \wedge U\vec{z}_{U,V} \not\uparrow_\iota V\vec{z}_{U,V}$,

*where $\vec{z}_{U,V} = z_{U,V,1}, \ldots, z_{U,V,p}$ and $z_{U,V,i} = \{\, a \mid \mathrm{Sep}^i_\rho(U, V, a) \,\}$.*

Proof. By induction on $\rho$.

# Proof of the density theorem. Case $\rho \to \sigma$, (a)

- Fix $W = \{ (U_i, a_i) \mid i < n \} \in \mathrm{Con}_{\rho\to\sigma}$. Consider $i < j < n$ with $a_i \not\uparrow a_j$, thus $U_i \not\uparrow U_j$.
- By IH(b) for $\rho$ we have $\vec{z}_{ij} \in G$ such that $U_i\vec{z}_{ij} \not\uparrow_\iota U_j\vec{z}_{ij}$.
- Define for every $U \in \mathrm{Con}_\rho$ a set $I_U$ of indices $k < n$ such that "$U$ behaves as $U_k$ with respect to the $\vec{z}_{ij}$":

$$I_U := \{ k < n \mid \forall_{i<k}(a_i \not\uparrow a_k \to U\vec{z}_{ik} \vdash_\iota U_k\vec{z}_{ik}) \wedge \forall_{j>k}(a_k \not\uparrow a_j \to U\vec{z}_{kj} \vdash_\iota U_k\vec{z}_{kj}) \}.$$

Notice that $k \in I_{U_k}$.

# Case $\rho \to \sigma$, (a) (continued)

Recall

$$I_U := \{\, k < n \mid \forall_{i<k}(a_i \not\uparrow a_k \to U\vec{z}_{ik} \vdash_\iota U_k\vec{z}_{ik}) \land \forall_{j>k}(a_k \not\uparrow a_j \to U\vec{z}_{kj} \vdash_\iota U_k\vec{z}_{kj}) \,\}.$$

We show $V_U := \{\, a_k \mid k \in I_U \,\} \in \mathrm{Con}_\sigma$.

- It suffices to prove $a_i \uparrow a_j$ for $i, j \in I_U$ with $i < j$.
- Since $a_i \uparrow a_j$ is decidable we can argue indirectly: let $a_i \not\uparrow a_j$.
- Then $U\vec{z}_{ij} \vdash_\iota U_j\vec{z}_{ij}$ and $U\vec{z}_{ij} \vdash_\iota U_i\vec{z}_{ij}$, thus $U_i\vec{z}_{ij} \uparrow_\iota U_j\vec{z}_{ij}$.
- But $U_i\vec{z}_{ij} \not\uparrow_\iota U_j\vec{z}_{ij}$ by the choice of the $\vec{z}_{ij}$ for $U_i \not\uparrow U_j$.

$$I_U := \{\, k < n \mid \forall_{i<k}(a_i \not\uparrow a_k \to U\vec{z}_{ik} \vdash_\iota U_k\vec{z}_{ik}) \wedge \forall_{j>k}(a_k \not\uparrow a_j \to U\vec{z}_{kj} \vdash_\iota U_k\vec{z}_{kj}) \,\}.$$

By IH(a) for $\sigma$, $V_U \subseteq y_{V_U} := \{\, a \mid \mathrm{TExt}_\sigma(V_U, a) \,\} \in G_\sigma$. Let

$$f := \{\, (U, a) \mid [a \in y_{V_U} \wedge \forall_{i,j<n}(a_i \not\uparrow a_j \to G_\iota(U\vec{z}_{ij}))] \vee V_U \vdash a \,\}$$

Claim: $W \subseteq f \in G_{\rho\to\sigma}$ (then $\mathrm{TExt}_{\rho\to\sigma}(W, (U, a)) := f$-fla).

- Since $k \in I_{U_k}$ we have $a_k \in V_{U_k}$. Thus $(U_k, a_k) \in f$.
- For $\mathrm{Ideal}_{\rho\to\sigma}(f)$ verify the properties of "approximable maps".
- Here only: $(U, a) \in f$ and $U' \vdash U$ imply $(U', a) \in f$. By cases.
- *Case* [...]. Have $a \in y_{V_U}$ and $\forall_{i,j<n}(a_i \not\uparrow a_j \to G_\iota(U'\vec{z}_{ij}))$. Show $a \in y_{V_{U'}}$. For $a_i \not\uparrow a_j$ both $U\vec{z}_{ij}$ and $U'\vec{z}_{ij}$ have total $a$. The same, since $U' \vdash U$. Thus $I_U = I_{U'}$, hence $V_U = V_{U'}$.
- *Case* $V_U \vdash a$. $U' \vdash U$ implies $I_U \subseteq I_{U'}$, hence $V_U \subseteq V_{U'}$, hence $V_{U'} \vdash a$ and therefore $(U', a) \in f$.

# Case $\rho \to \sigma$, (a) (continued)

Recall

$$f := \{ (U, a) \mid [a \in y_{V_U} \wedge \forall_{i,j<n}(a_i \not\uparrow a_j \to G_\iota(U\vec{z}_{ij}))] \vee V_U \vdash a \}.$$

It remains to prove $f \in G_{\rho\to\sigma}$. Let $x \in G_\rho$.

- We show $fx \in G_\sigma$, i.e., $\{ a \mid \exists_{U \subseteq x}((U, a) \in f) \} \in G_\sigma$.
- Recall $\vec{z}_{ij} \in G$ for all $i < j < n$ with $a_i \not\uparrow a_j$. Hence $x\vec{z}_{ij} \in G_\iota$.
- $U_{ij} \subseteq x$ with $G_\iota(U_{ij}\vec{z}_{ij})$. Same total $a$ in all $U_{ij}\vec{z}_{ij}$.
- Let $U$ be the union of all $U_{ij}$'s. Then $G_\iota(U\vec{z}_{ij})$.
- $(U, a) \in f$ for all $a \in y_{V_U}$, i.e., $y_{V_U} \subseteq fx$ and thus $fx \in G_\sigma$.

# Proof of the density theorem. Case $\rho \to \sigma$, (b)

- Let $W_1, W_2 \in \mathrm{Con}_{\rho\to\sigma}$ with $W_1 \not\uparrow W_2$. Take the first $(U_i, a_i) \in W_i$ such that $U_1 \uparrow U_2$ and $a_1 \not\uparrow a_2$.
- By IH(a) for $\rho$

$$U_1 \cup U_2 \subseteq z_{U_1,U_2} := \{\, a \mid \mathrm{TExt}_\rho(U_1 \cup U_2, a) \,\} \in G_\rho.$$

Thus $a_i \in W_i z_{U_1,U_2}$.

- IH(b) for $\sigma$ gives $\vec{z}_{a_1,a_2} \in G$ such that

$$\{a_1\}\vec{z}_{a_1,a_2} \not\uparrow_\iota \{a_2\}\vec{z}_{a_1,a_2}, \quad z_{a_1,a_2,i} := \{\, a \mid \mathrm{Sep}^i_\sigma(\{a_1\},\{a_2\},a) \,\}.$$

- Hence $W_1 z_{U_1,U_2}\vec{z}_{a_1,a_2} \not\uparrow_\iota W_2 z_{U_1,U_2}\vec{z}_{a_1,a_2}$. Thus define

$$\mathrm{Sep}^1_{\rho\to\sigma}(W_1, W_2, a) := \mathrm{TExt}_\rho(U_1 \cup U_2, a),$$
$$\mathrm{Sep}^{i+1}_{\rho\to\sigma}(W_1, W_2, a) := \mathrm{Sep}^i_\sigma(\{a_1\},\{a_2\},a).$$

# Conclusion

- Basic semantical concept: Partial continuous functionals.
- Ideal (or point): a consistent deductively closed set of tokens.
- $\mathrm{TCF}$: Theory of computable functionals. Types and formulas kept separate.
- $\mathrm{TCF}^+$: "pointfree" refinement. Functionals appear as ideals.
- Sketch of proof of the density theorem in $\mathrm{TCF}^+$.

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