Equality and extensionality

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- Goal: a type theory allowing infinite data.
- Reason: real numbers are best represented as streams (i.e., possibly infinite lists) of signed digits, or else using Gray code (U. Berger, Di Gianantonio, Miyamoto, Tsuiki, Wiesnet).
- New: treatment of extensionality, similar to Gandy "On the axiom of extensionality – part I", JSL 1956.

Constructor types κ have the form

$$\vec{\alpha} \to (\xi)_{i<n} \to \xi$$

with all type variables α_i distinct from each other and from ξ. We call

$$\iota := \mu_\xi \vec{\kappa}$$

an algebra form

Examples

Algebra forms without parameter type variables are

$$
\begin{aligned}
\mathbb{U} &:= \mu_\xi \xi && \text{(unit)},\\
\mathbb{B} &:= \mu_\xi(\xi, \xi) && \text{(booleans)},\\
\mathbb{N} &:= \mu_\xi(\xi, \xi \to \xi) && \text{(natural numbers, unary)},\\
\mathbb{P} &:= \mu_\xi(\xi, \xi \to \xi, \xi \to \xi) && \text{(positive numbers, binary)},\\
\mathbb{D} &:= \mu_\xi(\xi, \xi \to \xi \to \xi) && \text{(binary trees, or derivations)}.
\end{aligned}
$$

Algebra forms with type parameters are

$$
\begin{aligned}
\mathbb{I}(\alpha) &:= \mu_\xi(\alpha \to \xi) && \text{(identity)},\\
\mathbb{L}(\alpha) &:= \mu_\xi(\xi, \alpha \to \xi \to \xi) && \text{(lists)},\\
\mathbb{S}(\alpha) &:= \mu_\xi(\alpha \to \xi \to \xi) && \text{(streams)},\\
\mathbb{L}^+(\alpha, \beta) &:= \mu_\xi(\alpha \to \xi, \beta \to \xi \to \xi) && \text{(non-empty lists)}\\
\alpha \times \beta &:= \mu_\xi(\alpha \to \beta \to \xi) && \text{(product)},\\
\alpha + \beta &:= \mu_\xi(\alpha \to \xi, \beta \to \xi) && \text{(sum)}.
\end{aligned}
$$

Types are

$$\rho, \sigma, \tau ::= \alpha \mid \iota(\vec{\rho}) \mid \rho \to \sigma,$$

where ι is an algebra form with $\vec{\alpha}$ its parameter type variables, and $\iota(\vec{\rho})$ the result of substituting the (already generated) types $\vec{\rho}$. Types of the form $\iota(\vec{\rho})$ are algebras. Let $|\iota(\vec{\rho})| := 1 + \max|\vec{\rho}|$. The level of a type is defined by

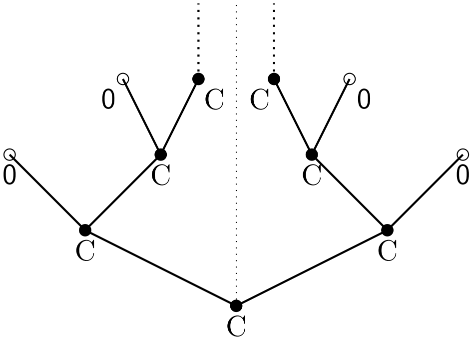
$$\begin{aligned} \mathrm{lev}(\alpha) &:= 0, \\ \mathrm{lev}(\iota(\vec{\rho})) &:= \max(\mathrm{lev}(\vec{\rho})), \\ \mathrm{lev}(\rho \to \sigma) &:= \max(\mathrm{lev}(\sigma), 1 + \mathrm{lev}(\rho)). \end{aligned}$$

Base types are types of level 0, and a higher type has level ≥ 1.

Examples. 1. $\mathbb{L}(\alpha)$, $\mathbb{L}(\mathbb{L}(\alpha))$, $\alpha \times \beta$ are algebras.
2. $\mathbb{L}(\mathbb{L}(\mathbb{N}))$, $\mathbb{Z} := \mathbb{P} + \mathbb{U} + \mathbb{P}$, $\mathbb{Q} := \mathbb{Z} \times \mathbb{P}$ are closed base types.
3. $\mathbb{R} := (\mathbb{N} \to \mathbb{Q}) \times (\mathbb{P} \to \mathbb{N})$ is a closed algebra of level 1.

Semantics

By x^ρ we denote "objects" of type ρ. They are "ideals" in the Scott-Ershov model of partial continuous functionals. Such objects can be infinite, already in closed base types.



This $x^{\mathbb{D}}$ is the "deductive closure" of the "tokens" t_n where

$$t_1 := \mathrm{C}{*}{*}, \qquad t_{n+1} := \mathrm{C} t_n t_n.$$

Syntax

Terms of T^+ are built from (typed) variables, constructors C or defined constants D by abstraction and application:

$$M, N ::= x^\rho \mid \mathrm{C}^\rho \mid D^\rho \mid (\lambda_{x^\rho} M^\sigma)^{\rho\to\sigma} \mid (M^{\rho\to\sigma} N^\rho)^\sigma.$$

Defined constants D come with a system of computation rules

$$D\vec{P}_i(\vec{y}_i) = M_i \qquad (i = 1, \dots, n)$$

Examples. 1. $\mathcal{R}^\tau_\mathbb{N} \colon \mathbb{N} \to \tau \to (\mathbb{N} \to \tau \to \tau) \to \tau$, with rules

$$\mathcal{R}^\tau_\mathbb{N} 0af = a, \qquad \mathcal{R}^\tau_\mathbb{N}(Sn)af = fn(\mathcal{R}^\tau_\mathbb{N} naf).$$

2. ${}^{\mathrm{co}}\mathcal{R}^\tau_\mathbb{N} \colon \tau \to (\tau \to \mathbb{U} + (\mathbb{N} + \tau)) \to \mathbb{N}$, with rules

$${}^{\mathrm{co}}\mathcal{R}^\tau_\mathbb{N} xf = \begin{cases} 0 & \text{if } fx \equiv \mathrm{DummyL}^{\mathbb{U}+(\mathbb{N}+\tau)} \\ Sn & \text{if } fx \equiv \mathrm{Inr}(\mathrm{InL}^{\mathbb{N}\to\mathbb{N}+\tau} n) \\ S({}^{\mathrm{co}}\mathcal{R}^\tau_\mathbb{N} x' f) & \text{if } fx \equiv \mathrm{Inr}(\mathrm{InR}^{\tau\to\mathbb{N}+\tau} x') \end{cases}$$

Clauses and Predicate Forms

Assume an infinite supply of predicate variables, each of its own arity (a list of types). Distinguish "computationally relevant" ones $X \dots$ and "non-computational" ones $X^{\mathrm{nc}} \dots$. By $\bar{X}$ or $\bar{X}^{\mathrm{nc}}$ we denote the result of applying X or X^{nc} to a list of terms of fitting types, and by $\tilde{X}$ or $\tilde{X}^{\mathrm{nc}}$ lists of those.

Clauses K have the form

$$\forall_{\vec{x}}(\tilde{Y} \to \tilde{Z}^{\mathrm{nc}} \to (\forall_{\vec{y}_i}(\tilde{W}_i^{\mathrm{nc}} \to \bar{X}_i))_{i<n} \to \bar{X})$$

with all predicate variables Y_i, Z_i^{nc}, W_i^{nc} occuring exactly once and distinct from each other and from X. Predicate forms are

$$I := (\mu/\nu)_X \vec{K}, \qquad I^{\mathrm{nc}} := (\mu^{\mathrm{nc}}/\nu^{\mathrm{nc}})_X \vec{K}$$

Predicates and Formulas

Definition (Predicates and formulas)

$$
\begin{aligned}
P, Q &::= X \mid X^{\mathrm{nc}} \mid \{\, \vec{x} \mid A \,\} \mid I(\vec{\rho}, \vec{P}, \vec{Q}) \mid I^{\mathrm{nc}}(\vec{\rho}, \vec{P}),\\
A, B &::= P\vec{t} \mid A \to B \mid \forall_x A
\end{aligned}
$$

with I and I^{nc} predicate forms.

Totality and Cototality

For closed base types $\iota(\vec{\rho})$ define (co)totality predicates $T_{\iota,\vec{\rho}}$, ${}^{\mathrm{co}}T_{\iota,\vec{\rho}}$ of arity $(\iota(\vec{\rho}))$ by induction on $|\iota(\vec{\rho})| := 1 + \max |\vec{\rho}|$. Here ι is an algebra form (e.g. $\mathbb{L}$). $\vec{\rho}$ are closed base types.

Examples. (i). $T_{\mathbb{N}} := \mu_X(K_0, K_1)$ and ${}^{\mathrm{co}}T_{\mathbb{N}} := \nu_X(K_0, K_1)$ with

$$K_0 := (0 \in X)$$
$$K_1 := \forall_n (n \in X \to Sn \in X)$$

(ii). $T_{\mathbb{L},\mathbb{N}} := \mu_X(K_0, K_1)$ and ${}^{\mathrm{co}}T_{\mathbb{L},\mathbb{N}} := \nu_X(K_0, K_1)$ with clauses

$$K_0 := ([] \in X)$$
$$K_1 := \forall_{n,l} (n \in T_{\mathbb{N}} \to l \in X \to n :: l \in X)$$

(iii). $T_{\mathbb{L},\mathbb{L}(\mathbb{N})} := \mu_X(K_0, K_1)$ and ${}^{\mathrm{co}}T_{\mathbb{L},\mathbb{L}(\mathbb{N})} := \nu_X(K_0, K_1)$ with

$$K_0 := ([] \in X)$$
$$K_1 := \forall_{l,u} (l \in T_{\mathbb{L},\mathbb{N}} \to u \in X \to l :: u \in X)$$

For every algebra form ι with parameters $\vec{\alpha}$ we define two predicate forms: similarity $\sim_\iota$ and bisimilarity $\approx_\iota$ with parameters $\vec{\alpha}$, $\vec{Y}$ (where Y_i has arity (α_i, α_i) for each α_i).

Let $\vec{\alpha} \to (\xi)_{i<n} \to \xi$ be a constructor type. Take $(\mu/\nu)_Z(\vec{K})$, where the clause corresponding to the constructor type above is

$$Y_1 u_1 u_1' \to \dots Y_n u_n u_n' \to Z_1 v_1 v_1' \to \dots Z_m v_m v_m' \to Z(\mathrm{C}\vec{u}\vec{v}, \mathrm{C}\vec{u}\,'\vec{v}\,'))$$

with C the corresponding constructor of ι.

Example: The constructor types of $\mathbb{L}(\alpha)$ are ξ and $\alpha \to \xi \to \xi$, and the corresponding clauses are

$$\begin{aligned} &K_0 \colon Z([]_\alpha, []_\alpha), \\ &K_1 \colon \forall_{x,x',u,u'}(Yxx' \to Zuu' \to Z(x :: u, x' :: u')). \end{aligned}$$

$\sim := \mu_Z(K_0, K_1)$ is the least fixed point of these two clauses.
$\approx := \nu_Z(K_0, K_1)$ is the greatest fixed point of the closure axiom

$$\begin{aligned} (u \approx u') \to {} & (u \equiv []_\alpha \wedge u' \equiv []_\alpha) \vee \\ & \exists_{x,u_1,x',u_1'}(Yxx' \wedge u_1 \approx u_1' \wedge u \equiv x :: u_1 \wedge u' \equiv x' :: u_1'). \end{aligned}$$

Definition (Pattern $\|C\|$ of a predicate or formula C)

For C n.c. let $\|C\| := 0$. Assume C is c.r.

$$\begin{aligned}
\|(I/^{\mathrm{co}}I)(\vec{\rho}, \vec{P}, \vec{Q})\| &:= (I/^{\mathrm{co}}I)(\|\vec{P}\|) \\
\|P\vec{t}\,\| &:= \|P\| \\
\|A \to B\| &:= \begin{cases} \|A\| \to \|B\| & \text{for } A \text{ c.r.} \\ \|B\| & \text{otherwise} \end{cases} \\
\|\forall_x A\| &:= \|A\|
\end{aligned}$$

Definition (Type $\tau(U)$ of a c.r. predicate pattern U)

$$\begin{aligned}
\tau((I/^{\mathrm{co}}I)(\vec{U})) &:= \iota(\tau(\vec{U})) \\
\tau(U \to V) &:= \begin{cases} \tau(U) \to \tau(V) & \text{for } U \text{ c.r.} \\ \tau(V) & \text{otherwise} \end{cases}
\end{aligned}$$

Here ι is the name of the algebra associated to $(I/^{\mathrm{co}}I)$.

Equality and Extensionality

Definition (Predicates $\doteq_U$ and Ext_U)

$$
\begin{aligned}
(x \doteq_{(I/^{\mathrm{co}} I)(\vec{U}\,)} y) &:= (x\,(\sim / \approx)_{\iota, \doteq_{\vec{U}}}\, y)\\
(x \in \mathrm{Ext}_{(I/^{\mathrm{co}} I)(\vec{U}\,)}) &:= (x \in (T/^{\mathrm{co}}\, T)_{\iota, \mathrm{Ext}_{\vec{U}}})\\
(f \doteq_{U \to V} g) &:= \begin{cases} f \doteq_V g & \text{if } U = 0\\ \forall_{x \in (T/^{\mathrm{co}}\, T)_{\iota,\vec{\rho}}} (fx \doteq_V gx) & \text{if } U = (I/^{\mathrm{co}} I)(\vec{U}\,),\\ & \tau(U) \text{ given by } \iota, \vec{\rho}\\ \forall_{x,y} (x \doteq_U y \to fx \doteq_V gy) \end{cases}\\
(f \in \mathrm{Ext}_{U \to V}) &:= (f \doteq_{U \to V} f)
\end{aligned}
$$

Definition ($C^{\mathbf{r}}$ for predicates and formulas C)

For n.c. C let $C^{\mathbf{r}} := C$. If C is c.r. $C^{\mathbf{r}}$ is a predicate of arity $(\vec{\sigma}, \tau(C))$ with $\vec{\sigma}$ the arity of C. Write $z \mathbf{\ r\ } C$ for $C^{\mathbf{r}}z$ if C is a c.r. formula. For c.r. predicates let $X^{\mathbf{r}}$ be an n.c. predicate variable, and

$$\{\, \vec{x} \mid A \,\}^{\mathbf{r}} := \{\, \vec{x}, z \mid z \mathbf{\ r\ } A \,\}.$$

For a c.r. predicate form

$$I := (\mu/\nu)_X (\forall_{\vec{x}_i}(\tilde{Y}_i \to \tilde{Z}_i^{\mathrm{nc}} \to (\forall_{\vec{y}_{i\nu}}(\tilde{W}_{i\nu}^{\mathrm{nc}} \to \bar{X}_{i\nu}))_{\nu<n_i} \to \bar{X}_i))_{i<k}$$

we define the n.c. witnessing predicate form

$$I^{\mathbf{r}} := (\mu/\nu)_{X^{\mathbf{r}}} (\forall_{\vec{x}_i, \vec{u}_i, \vec{v}_i}(\vec{u}_i \mathbf{\ r\ } \tilde{Y}_i \to \tilde{Z}_i^{\mathrm{nc}} \to (\forall_{\vec{y}_{i\nu}}(\tilde{W}_{i\nu}^{\mathrm{nc}} \to v_{i\nu} \mathbf{\ r\ } \bar{X}_{i\nu}))_{\nu<n_i} \to \mathrm{C}_i \vec{u}_i \vec{v}_i \mathbf{\ r\ } \bar{X}))_{i<k}$$

Here C_i is the i-th constructor of the algebra form ι_I with constructor types $\tau(K_i)$, K_i the i-th clause of I. For a c.r. inductive predicate $I(\vec{\rho}, \vec{P}, \vec{Q}\,)$ we define $I(\vec{\rho}, \vec{P}, \vec{Q}\,)^{\mathbf{r}}$ to be $I^{\mathbf{r}}(\vec{\rho}, \vec{P}^{\mathbf{r}}, \vec{Q}\,)$.

$C^{\mathbf{r}}$ for predicates and formulas C (continued)

For c.r. *formulas* let

$$
\begin{aligned}
z \;\mathbf{r}\; P\vec{t} &:= P^{\mathbf{r}}\vec{t}z, \\
z \;\mathbf{r}\; (A \to B) &:= \begin{cases} \forall_w (w \;\mathbf{r}\; A \to zw \;\mathbf{r}\; B) & \text{if } A \text{ is c.r.} \\ A \to z \;\mathbf{r}\; B & \text{if } A \text{ is n.c.} \end{cases} \\
z \;\mathbf{r}\; \forall_x A &:= \forall_x (z \;\mathbf{r}\; A).
\end{aligned}
$$

- According to Kolmogorov (1932) a c.r. formula A should be viewed as a "computational problem", asking for a solution.
- This solution should be a functional of type $\tau(A)$ which is "mathematically reasonable", i.e. extensional w.r.t. A.

We express this view in the form of invariance axioms:

$$\mathrm{Inv}_A \colon A \leftrightarrow \exists_{z \in \mathrm{Ext}_A}(z \mathbf{~r~} A).$$

Extracted term $\mathrm{et}(M)$ of a derivation M^A with A c.r.

$$
\begin{aligned}
\mathrm{et}(u^A) &:= z_u^{\tau(A)} \quad (z_u^{\tau(A)} \text{ uniquely associated to } u^A),\\
\mathrm{et}((\lambda_{u^A} M^B)^{A\to B}) &:= \begin{cases} \lambda_{z_u}^{\tau(A)}\mathrm{et}(M) & \text{if } A \text{ is c.r.}\\ \mathrm{et}(M) & \text{if } A \text{ is n.c.}\end{cases}\\
\mathrm{et}((M^{A\to B} N^A)^B) &:= \begin{cases} \mathrm{et}(M)\mathrm{et}(N) & \text{if } A \text{ is c.r.}\\ \mathrm{et}(M) & \text{if } A \text{ is n.c.}\end{cases}\\
\mathrm{et}((\lambda_x M^A)^{\forall_x A}) &:= \mathrm{et}(M),\\
\mathrm{et}((M^{\forall_x A(x)} r)^{A(r)}) &:= \mathrm{et}(M).
\end{aligned}
$$

Extracted term $\mathrm{et}(M)$ of a derivation M^A (ctd.)

It remains to define extracted terms for the axioms. Consider a (c.r.) inductively defined predicate I. For its introduction and elimination axioms define

$$\mathrm{et}(I_i^+) := \mathrm{C}_i \quad \text{and} \quad \mathrm{et}(I^-) := \mathcal{R},$$

where both the constructor C_i and the recursion operator $\mathcal{R}$ refer to the algebra ι_I associated with I. For the closure and greatest-fixed-point axioms of ${}^{\mathrm{co}}I$ define

$$\mathrm{et}({}^{\mathrm{co}}I_i^+) := {}^{\mathrm{co}}\mathcal{R} \quad \text{and} \quad \mathrm{et}({}^{\mathrm{co}}I^-) := \mathrm{D},$$

where both the corecursion operator ${}^{\mathrm{co}}\mathcal{R}$ and the destructor D refer again to the algebra ι_I associated with I. For the invariance axioms we take the respective identities.

Lemma (Extensionality of axiom-free proof terms)

For every proof $M: A$ without axioms and with free assumptions among $\vec{u}: \vec{C}$ we have

$$z_{\vec{u}}^{\tau(\vec{C})} \in \mathrm{Ext}_{\vec{C}} \to \mathrm{et}(M)^{\tau(A)} \in \mathrm{Ext}_A.$$

Lemma (Extensionality of the recursion operator)

Let I be an inductive predicate and ι_I its associated algebra. Then the extracted term $\mathrm{et}(I^-) := \mathcal{R}_{\iota_I}^{\tau}$ of its least-fixed-point (or elimination) axiom I^- is extensional w.r.t. I^-

Lemma (Extensionality of the corecursion operator)

Let ${}^{\mathrm{co}}I$ be a coinductive predicate and ι_I its associated algebra. Then the extracted term $\mathrm{et}({}^{\mathrm{co}}I^+) := {}^{\mathrm{co}}\mathcal{R}_{\iota_I}^{\tau}$ of its greatest-fixed-point (or coinduction) axiom ${}^{\mathrm{co}}I^+$ is extensional w.r.t. ${}^{\mathrm{co}}I^+$.

Theorem (Soundness)

Let M be a derivation of a formula A from assumptions $u_i\colon C_i$ $(i < n)$. Then we can derive

$$\begin{cases} \mathrm{et}(M) \in \mathrm{Ext}_A,\ \mathrm{et}(M) \mathrel{\mathbf{r}} A & \textit{if } A \textit{ is c.r.} \\ A & \textit{if } A \textit{ is n.c.} \end{cases}$$

from assumptions

$$\begin{cases} z_{u_i} \in \mathrm{Ext}_{C_i},\ z_{u_i} \mathrel{\mathbf{r}} C_i & \textit{if } C_i \textit{ is c.r.} \\ C_i & \textit{if } C_i \textit{ is n.c.} \end{cases}$$