Embedding classical in minimal implicational logic

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Context and notation

- $A, B, \ldots$ formulas of implicational (propositional) logic, built from propositional variables $P, Q, \ldots$ by implication $\to$.
- $\neg A := A \to \bot$ and $\neg_* A := A \to *$.
- $\vdash_c$ and $\vdash_i$ denote classical and intuitionistic derivability.
- $\vdash_c A$ means $\mathrm{Stab}_{\mathcal{V}(A)} \vdash A$ and $\vdash_i A$ means $\mathrm{Efq}_{\mathcal{V}(A)} \vdash A$, where $\vdash$ denotes derivability in minimal logic,

$$\begin{aligned} \mathrm{Stab}_V &:= \{\, \neg\neg P \to P \mid P \in V \,\}, \\ \mathrm{Efq}_V &:= \ \{\, \bot \to P \mid P \in V \,\}. \end{aligned}$$

Assume $\vdash_c A$.

- Which assumptions on the propositional variables P in A are needed for $\vdash_i A$?
- Ishihara 2014: $\Delta \vdash_i A$ for Δ a set of disjunctions $P \vee \neg P$.
- Here: Instead of $P \vee \neg P$ we take

$$\begin{aligned} \mathrm{Stab}_P &: \quad \neg\neg P \to P \\ \mathrm{Peirce}_{Q,P} &: ((Q \to P) \to Q) \to Q \end{aligned}$$

Results

- $\vdash_c A$ implies $\mathrm{Stab}_P \vdash_i A$ for P the final conclusion of A.
- $\vdash_c A$ implies $\Pi_A \vdash A$ for

$$\begin{aligned}\Pi_A := \{\, & \mathrm{Peirce}_{*,P} \mid \\ & P \text{ final conclusion of a positive subformula of } A\,\} \\ \cup\, & \{\bot \to *\}\end{aligned}$$

with $*$ a new prop. variable and $\bot \to *$ present only if $\bot$ in A.

- Intuitionistic logic and stability
- Minimal logic and Peirce formulas
- Examples

Work in Gentzen's natural deduction calculus.

Proposition.

(a) $\Gamma \vdash_c A$ implies $\mathrm{Stab}_*, \neg_*\neg\Gamma \vdash_i \neg_*\neg_* A$.

(b) $\Gamma \vdash_c A$ implies $\mathrm{Stab}_*, \Gamma \vdash_i \neg_*\neg_* A$.

Proof of (b) from (a).

Note that $\vdash (\perp \to *) \to A \to \neg_*\neg A$. But $\perp \to *$ is a consequence of Stab_*. $\square$

Proof of (a) $\Gamma \vdash_c A$ implies $\mathrm{Stab}_*, \neg_*\neg\Gamma \vdash_i \neg_*\neg_* A$

By induction on $\Gamma \vdash_c A$.

Case Ax. Since our only axiom is stability $\neg\neg A \to A$ we must prove $\mathrm{Stab}_* \vdash_i \neg_*\neg_*(\neg\neg A \to A)$.

It is easiest to find such a proof with the help of a proof assistant (`http://www.minlog-system.de`, writing `F` for $\perp$ and `S` for $*$):

$\mathrm{Stab}_* \vdash_i \neg_* \neg_* (\neg\neg A \to A)$

```
u: F -> A
u0: ((S -> F) -> F) -> S
u1: (((A -> F) -> F) -> A) -> S
u2: S -> F
u3: (A -> F) -> F
u4: S -> F
u5: A
u6: (A -> F) -> F
(lambda (u)
 (lambda (u0)
  (lambda (u1)
   (u0 (lambda (u2)
        (u2 (u1 (lambda (u3)
                 (u (u2 (u0 (lambda (u4)
                             (u3 (lambda (u5)
                                  (u2 (u1 (lambda (u6)
                                           u5))...)
```

Proof of (a) $\Gamma \vdash_c A$ implies $\mathrm{Stab}_*, \neg_*\neg\Gamma \vdash_i \neg_*\neg_* A$

Use

$$\vdash (\neg\neg * \to *) \to \neg_*\neg A \to \neg_*\neg_* A, \tag{1}$$
$$\vdash (\bot \to B) \to (\neg_*\neg A \to \neg_*\neg_* B) \to \neg_*\neg_*(A \to B). \tag{2}$$

Case Assumption. Goal: $\mathrm{Stab}_*, \neg_*\neg A \vdash_i \neg_*\neg_* A$. Follows from (**??**). *Case* $\to^+$.

$$\dfrac{\begin{array}{c}[u\colon A] \\ \mid M \\ B\end{array}}{A \to B}\to^+ u$$

By induction hypothesis

$$\mathrm{Stab}_*, \neg_*\neg\Gamma, \neg_*\neg A \vdash_i \neg_*\neg_* B.$$

The claim $\mathrm{Stab}_*, \neg_*\neg\Gamma \vdash_i \neg_*\neg_*(A \to B)$ follows from (**??**).

One instance of stability suffices

Theorem

$\vdash_c A$ implies $\mathrm{Stab}_P \vdash_i A$ for P the final conclusion of A.

Proof.

Let $A = \Gamma \to P$. Recall

(b) $\Gamma \vdash_c P$ implies $\mathrm{Stab}_*, \Gamma \vdash_i \neg_* \neg_* P$.

Hence

$$\mathrm{Stab}_*, \Gamma, \neg_* P \vdash_i *$$

with $*$ new. Substituting $*$ by P gives $\mathrm{Stab}_P, \Gamma, P \to P \vdash_i P$. $\square$

Glivenko's theorem

says that every negation proved classically can also be proved intuitionistically.

Corollary (Glivenko).
$\Gamma \vdash_c \bot$ implies $\Gamma \vdash_i \bot$.

Proof. In the theorem let $A = \Gamma \to \bot$:

$$\Gamma \vdash_c \bot \text{ implies } \mathrm{Stab}_\bot, \Gamma \vdash_i \bot.$$

But $\mathrm{Stab}_\bot$ is $((\bot \to \bot) \to \bot) \to \bot$ and hence easy to prove.

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Use

- Peirce suffices for the final atom:

$$\vdash \mathrm{Peirce}_{*,B} \to \mathrm{Peirce}_{*,A\to B}.$$

- Double negation shift for $\to$ $(\mathrm{DNS}_{\to})$

$$\vdash \mathrm{Peirce}_{*,B} \to (A \to \neg_*\neg_* B) \to \neg_*\neg_*(A \to B).$$

- Work in Gentzen's **G3cp**.
- Let Γ, Δ denote multisets of implicational formulas.

By induction on derivations $\mathcal{D}\colon \Gamma \Rightarrow \Delta$ in **G3cp** we define $\Pi(\mathcal{D})$.

$\Pi(\mathcal{D})$ will be a set of formulas $\mathrm{Peirce}_{*,P}$ for P the final conclusion of a positive subformula of $\Gamma \Rightarrow \Delta$, plus possibly (depending on which axioms appear in $\mathcal{D}$) the formula $\bot \to *$.

- *Cases* Ax: $P, \Gamma \Rightarrow \Delta, P$ and $\mathrm{L}\bot$: $\bot, \Gamma \Rightarrow \Delta$. We can assume that Γ and Δ are atomic. If $\Gamma \cap \Delta = \emptyset$ let $\Pi(\mathcal{D}) := \{\bot \to *\}$, and $:= \emptyset$ otherwise.
- *Case* $\mathrm{L}{\to}$. Then $\mathcal{D}$ ends with

$$\dfrac{\begin{array}{c} \mid \mathcal{D}_1 \\ \Gamma \Rightarrow \Delta, A \end{array} \qquad \begin{array}{c} \mid \mathcal{D}_2 \\ B, \Gamma \Rightarrow \Delta \end{array}}{A \to B, \Gamma \Rightarrow \Delta}\ \mathrm{L}{\to}$$

  Let $\Pi(\mathcal{D}) := \Pi(\mathcal{D}_1) \cup \Pi(\mathcal{D}_2)$.
- *Case* $\mathrm{R}{\to}$. Then $\mathcal{D}$ ends with

$$\dfrac{\begin{array}{c} \mid \mathcal{D}_1 \\ A, \Gamma \Rightarrow \Delta, B \end{array}}{\Gamma \Rightarrow \Delta, A \to B}\ \mathrm{R}{\to}$$

  Let $\Pi(\mathcal{D}) := \Pi(\mathcal{D}_1) \cup \{\mathrm{Peirce}_{*,P}\}$ (P final conclusion of B).

Proposition.

(a) Let $\mathcal{D}\colon \Gamma \Rightarrow \Delta$ in **G3cp**. Then $\vdash \Pi(\mathcal{D}), \Gamma, \neg_* \Delta \Rightarrow *$.

(b) Let $\mathcal{D}\colon \Gamma \Rightarrow *$ in **G3cp**. Then $\vdash \Pi(\mathcal{D}), \Gamma \Rightarrow *$.

Proof. (a). By induction on the derivation $\mathcal{D}$.
Case $\mathrm{L}\bot$. Then $\mathcal{D}\colon \bot, \Gamma \Rightarrow \Delta$ with Γ, Δ atomic. If $(\bot, \Gamma) \cap \Delta = \emptyset$ then $\Pi(\mathcal{D}) = \{\bot \to *\}$ and hence $\vdash \Pi(\mathcal{D}), \bot, \Gamma, \neg_* \Delta \Rightarrow *$.
Case $\mathrm{R}{\to}$. Then $\mathcal{D}$ ends with

$$\dfrac{\begin{array}{c} \mid \mathcal{D}_1 \\ A, \Gamma \Rightarrow \Delta, B \end{array}}{\Gamma \Rightarrow \Delta, A \to B}\,\mathrm{R}{\to}$$

$$\begin{array}{ll} \vdash \Pi(\mathcal{D}_1), \Gamma, \neg_* \Delta \Rightarrow A \to \neg_* \neg_* B & \text{by IH} \\ \vdash \mathrm{Peirce}_{*,B}, \Pi(\mathcal{D}_1), \Gamma, \neg_* \Delta \Rightarrow \neg_* \neg_* (A \to B) & \text{by } \mathrm{DNS}_{\to} \\ \vdash \Pi(\mathcal{D}), \Gamma, \neg_* \Delta, \neg_* (A \to B) \Rightarrow *. & \end{array}$$

Theorem.
$\vdash_c A$ implies $\Pi_A \vdash A$ for

$$\begin{aligned}\Pi_A := \{\, &\mathrm{Peirce}_{*,P} \mid \\ &P \text{ final conclusion of a positive subformula of } A \,\} \\ &\cup \{\bot \to *\}\end{aligned}$$

with $\bot \to *$ present only if $\bot$ in A.

Proof. **G3cp** is cut free, hence has the subformula property. Therefore a derivation in **G3cp** of a sequent without $\bot$ cannot involve $L\bot$. In this case $\Pi(\mathcal{D})$ consists of Peirce formulas only.

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Generalized Peirce formulas

$$\begin{aligned} A_0 :=& \quad (* \to P_0) \to * \\ A_{n+1} :=& (A_n \to P_{n+1}) \to * \end{aligned}$$

$$\mathrm{GP}_n := A_n \to *$$

For example

$$\begin{aligned} \mathrm{GP}_0 &= ((* \to P_0) \to *) \to * \\ \mathrm{GP}_1 &= ((((* \to P_0) \to *) \to P_1) \to *) \to * \\ \mathrm{GP}_2 &= ((((((* \to P_0) \to *) \to P_1) \to *) \to P_2) \to *) \to * \end{aligned}$$

Proposition.

(a) $(\mathrm{Peirce}_{*,P_i})_{i \le n} \vdash \mathrm{GP}_n$

(b) $(\mathrm{Peirce}_{*,P_i})_{i \le n, i \neq j} \nvdash \mathrm{GP}_n$.

Proof of (b). Assume $(\mathrm{Peirce}_{*,P_i})_{i \le n, i \neq j} \vdash \mathrm{GP}_n$. Substitute all P_i ($i \neq j$) by $*$. Then all Peirce_{*,P_i} ($i \neq j$) become provable and GP_n becomes equivalent to Peirce_{*,P_j}. Contradiction.

Example ($n = 2$, $j = 1$):

$$\mathrm{GP}_2 = ((((((* \to P_0) \to *) \to P_1) \to *) \to P_2) \to *) \to *$$

is turned into

$$((((((* \to *) \to *) \to P_1) \to *) \to *) \to *) \to *.$$

Examples where one Peirce formula suffices

Nagata formulas: another generalization of Peirce formulas.

$$\begin{aligned} N_0(A) &:= A \\ N_{k+1}(*, A_0, \ldots, A_k) &:= ((* \to N_k(A_0, \ldots, A_k)) \to *) \to *. \end{aligned}$$

For instance

$$\begin{aligned} N_1(*, A) &= ((* \to A) \to *) \to * \\ N_2(*, A, B) &= ((* \to N_1(A, B)) \to *) \to * \\ &= ((* \to ((A \to B) \to A) \to A) \to *) \to *. \end{aligned}$$

Examples where one Peirce formula suffices (continued)

Bull	$(((A \to B) \to B) \to *) \to ((A \to B) \to *) \to *$
Hosoi	$((B \to A) \to *) \to ((((A \to B) \to A) \to A) \to *) \to *$
Tarski	$(A \to *) \to ((A \to B) \to *) \to *$
Minari	$((* \to A) \to B) \to (B \to *) \to *$
Mints	$((((A \to B) \to A) \to A) \to *) \to *$
Glivenko	$(((B \to A) \to ((B \to C) \to A) \to A) \to *) \to *$