Constructive Analysis - Uniform Spaces in Minlog

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1 Finite Sequences

In this section, we introduce and prove properties of finite sequences in Setoids. We begin by introducing some basic notions such as functions and certain relations on setoids.

1.1 Basic Notions

1.1.1 Functions

A map $f : (U, \equiv_U) \to (V, \equiv_V)$ must satisfy:

$$\forall x \in U : f(x) \in V$$
$$\forall x, y \in U : x \equiv_U y \implies f(x) \equiv_V f(y)$$

1.1.2 Equivalnce Relations

An equivalence relation $\equiv_U$ on a setoid $(U, \equiv_U)$ must satisfy:

$$\forall x \in U : x \equiv_U x \qquad \text{(Reflexivity)}$$
$$\forall x, y \in U : x \equiv_U y \implies y \equiv_U x \qquad \text{(Symmetry)}$$
$$\forall x, y, z \in U : x \equiv_U y \wedge y \equiv_U z \implies x \equiv_U z \qquad \text{(Transitivity)}$$

1.1.3 Setoids

A setoid is a pair $(U, \equiv_U)$ where U is a set and $\equiv_U$ is an equivalence relation on U. In Minlog, we represent U as an unary predicate on some type, understood as the property of being an element of U. The equivalence relation $\equiv_U$ is a binary predicate on the same type. We will write $x \in U$ to express that x is an element of the setoid U and $x \in U^\tau$ to express that x is an element of the underlying type of the setoid U, but doesn't necessarily satisfy its predicate.

1.1.4 Preorder

A preorder $\leq_U$ on a setoid $(U, \equiv_U)$ must satisfy:

$$\forall x, y \in U : x \equiv_U y \implies x \leq_U y \qquad \text{(Reflexivity)}$$
$$\forall x, y, z \in U : x \leq_U y \land y \leq_U z \implies x \leq_U z \qquad \text{(Transitivity)}$$

1.1.5 Directed Set

A directed set is a pair $(I, \leq_I)$ with I a setoid and $\leq_I$ a preorder on I such that:

$$\forall i, j \in I : \exists k \in I : i \leq_I k \land j \leq_I k$$

1.2 Finite Sequences in Setoids

We will now define finite sequences in setoids. These are finite lists with some extra structure, namely that their elements are all from the same setoid. Given a setoid $(U, \equiv_U)$, we write U^* for the set (actually setoid) of finite sequences with elements from U.

1.2.1 Element Inclusion

We inductively define the SeqIn relation, written $\in_S$, which expresses that an element is included in a finite sequence. We define:

$$\forall x, y \in U^\tau : \forall s \in (U^*)^\tau : x \equiv_U y \implies x \in_S (y :: s)$$
$$\forall x, y \in U^\tau : \forall s \in (U^*)^\tau : x \in_S s \implies x \in_S (y :: s)$$

1.2.2 Properties of Element Inclusion

We can now prove some properties of the $\in_S$ relation.

SeqInNil $\forall x \in U^\tau : x \in_S [] \implies \bot$

SeqInStabAppd0 $\forall x \in U^\tau : \forall s, t \in (U^*)^\tau : x \in_S s \implies x \in_S (t +\!\!+ s)$

SeqInStabAppd1 $\forall x \in U^\tau : \forall s, t \in (U^*)^\tau : x \in_S t \implies x \in_S (t +\!\!+ s)$

1.2.3 Finit Sequences

A finite sequence $s \in U^*$ now must satisfy the following property:

$$\forall n \in \mathbb{N} : n < |s| \implies s_n \in U$$

We don't use $\in_S$ here, to avoid a circular definition.

1.2.4 Properties of Finite Sequences

We can now prove some properties of finite sequences.

FinSeqNil $[] \in U^*$

FinSeqElem $\forall x \in U^\tau : \forall s \in U^* : x \in_S s \implies x \in U$

FinSeqConsElem $\forall x \in U^\tau : \forall s \in (U^*)^\tau : (x :: s) \in U^* \implies x \in U$

FinSeqConsSub $\forall x \in U^\tau : \forall s \in (U^*)^\tau : (x :: s) \in U^* \implies s \in U^*$

FinSeqStabCons $\forall x \in U : \forall s \in U^* : (x :: s) \in U^*$

FinSeqStabAppd $\forall s, t \in U^* : (s \mathbin{++} t) \in U^*$

1.2.5 Sequence Equality

We want U^* to be a setoid, so we need to define an equivalence relation on it. We define the relation $\equiv_{U^*}$, written $\equiv_S$, as follows:

$$\forall s \in (U^*)^\tau : s \equiv_S s$$
$$\forall x, y \in U^\tau : \forall s, t \in (U^*)^\tau : x \equiv_U y \implies s \equiv_S t \implies (x :: s) \equiv_S (y :: t)$$

1.2.6 Properties of Sequence Equality

We can now prove some properties of the $\equiv_S$ relation.

SeqEqNilLeft $\forall s \in (U^*)^\tau : [] \equiv_S s \implies s = []$

SeqEqNilRight $\forall s \in (U^*)^\tau : s \equiv_S [] \implies s = []$

SeqEqConsElemEq $\forall x, y \in U^\tau : \forall s, t \in (U^*)^\tau : (x :: s) \equiv_S (y :: t)$
$\implies x \equiv_U y$

SeqEqConsSubEq $\forall x, y \in U^\tau : \forall s, t \in (U^*)^\tau : (x :: s) \equiv_S (y :: t)$
$\implies s \equiv_S t$

SeqEqLength $\forall s, t \in (U^*)^\tau : s \equiv_S t \implies |s| = |t|$

SeqEqConsNilLeft $\forall x \in U^\tau : \forall s \in (U^*)^\tau : [] \equiv_S (x :: s) \implies \bot$

SeqEqConsNilRight $\forall x \in U^\tau : \forall s \in (U^*)^\tau : (x :: s) \equiv_S [] \implies \bot$

SeqEqAppd $\forall s_0, s_1, t_0, t_1 \in (U^*)^\tau : s_0 \equiv_S t_0 \implies s_1 \equiv_S t_1$
$\implies (s_0 \mathbin{++} s_1) \equiv_S (t_0 \mathbin{++} t_1)$

SeqEqRefl $\forall s \in (U^*)^\tau : s \equiv_S s$

SeqEqSymm $\forall s, t \in (U^*)^\tau : t \in U^* \implies s \equiv_S t \implies t \equiv_S s$

SeqEqTrans $\forall s, t, u \in (U^*)^\tau : u \in U^* \implies s \equiv_S t \implies t \equiv_S u \implies s \equiv_S u$

SeqEqEqvRel $\equiv_S$ is an equivalence relation on U^*

SeqEqToPWEq $\forall s, t \in (U^*)^\tau : t \in U^* \implies s \equiv_S t \implies \forall n \in \mathbb{N} : n < |s| \implies s_n \equiv_U t_n$

SeqEqFinSeq $\forall s, t \in (U^*)^\tau : s \equiv_S t \implies s \in U^* \implies t \in U^*$

FinSeqEqSet U^* is a setoid with equivalence relation $\equiv_S$

1.2.7 Subsequences

Next, we define the notion of subsequences, written $\subseteq_S$. We define:

$$\forall s, t \in (U^*)^\tau : s = [] \implies s \subseteq_S t$$
$$\forall x \in U^\tau : \forall s, t \in (U^*)^\tau : x \in_S t \implies s \subseteq_S t \implies (x :: s) \subseteq_S t$$

1.2.8 Properties of Subsequences

We can now prove some properties of the $\subseteq_S$ relation.

SeqSubNilLeft $\forall s \in (U^*)^\tau : [] \subseteq_S s$

SeqSubNilRight $\forall s \in (U^*)^\tau : s \subseteq_S [] \implies s = []$

SeqSubNulRightCons $\forall x \in U^\tau : \forall s \in (U^*)^\tau : (x :: s) \subseteq_S [] \implies \bot$

SeqSubConsElem $\forall x \in U^\tau : \forall s, t \in (U^*)^\tau : (x :: s) \subseteq_S t \implies x \in_S t$

SeqSubConsSub $\forall x \in U^\tau : \forall s, t \in (U^*)^\tau : (x :: s) \subseteq_S t \implies s \subseteq_S t$

SeqSubStabCons0 $\forall s, t \in (U^*)^\tau : s \subseteq_S t \implies \forall x \in U^\tau : s \subseteq_S (x :: t)$

SeqSubStabCons1 $\forall s, t \in (U^*)^\tau : s \subseteq_S t \implies \forall x, y \in U^\tau : x \equiv_U y \implies (x :: s) \subseteq_S (y :: t)$

SeqSubStabAppd0 $\forall s, t_1 \in (U^*)^\tau : s \subseteq_S t_1 \implies \forall t_0 \in (U^*)^\tau : s \subseteq_S (t_0 ++ t_1)$

SeqSubStabAppd1 $\forall s, t_0 \in (U^*)^\tau : s \subseteq_S t_0 \implies \forall t_1 \in (U^*)^\tau : s \subseteq_S (t_0 ++ t_1)$

SeqSubStabAppd2 $\forall s_0, s_1, t \in (U^*)^\tau : s_0 \subseteq_S t \implies s_1 \subseteq_S t \implies (s_0 ++ s_1) \subseteq_S t$

SeqSubStabAppd $\forall s_0, s_1, t_0, t_1 \in (U^*)^\tau : s_0 \subseteq_S t_0 \implies s_1 \subseteq_S t_1 \implies (s_0 ++ s_1) \subseteq_S (t_0 ++ t_1)$

SeqSubRefl $\forall s, t \in (U^*)^\tau : t \in U^* \implies s \equiv_S t \implies s \subseteq_S t$

Using the properties of subsequences, we can prove some further properties of the $\in_S$ relation.

SeqInCompat $\forall s \in U^* : \forall x, y \in U^\tau : x \equiv_U y \implies x \in_S s \implies y \in_S s$

SeqInTrans $\forall x \in U^\tau : \forall s, t \in (U^*)^\tau : t \in U^* \implies s \subseteq_S t$
$\implies x \in_S s \implies x \in_S t$

This now allows us to show the following:

SeqSubTrans $\forall s, t, u \in (U^*)^\tau : u \in U^* \implies s \subseteq_S t \implies t \subseteq_S u \implies s \subseteq_S u$

1.2.9 Sequence Ordering

We now define a preorder on U^*, written $\leq_S$, as follows:

$$\forall s, t \in (U^*)^\tau : s \subseteq_S t \implies |s| \leq |t| \implies s \leq_S t$$

1.2.10 Properties of Sequence Ordering

We can now prove some properties of the $\leq_S$ relation.

SeqLeNilLeft $\forall s \in (U^*)^\tau : [] \leq_S s$

SeqLeNilRight $\forall s \in (U^*)^\tau : s \leq_S [] \implies s = []$

SeqLeNilRightCons $\forall x \in U^\tau : \forall s \in (U^*)^\tau : (x :: s) \leq_S [] \implies \bot$

SeqLeConsElem $\forall x \in U^\tau : \forall s, t \in (U^*)^\tau : (x :: s) \leq_S t \implies x \in_S t$

SeqLeConsLe $\forall x \in U^\tau : \forall s, t \in (U^*)^\tau : (x :: s) \leq_S t \implies s \leq_S t$

SeqLeStabCons0 $\forall s, t \in (U^*)^\tau : s \leq_S t \implies \forall x \in U^\tau : s \leq_S (x :: t)$

SeqLeStabCons1 $\forall s, t \in (U^*)^\tau : s \leq_S t \implies \forall x, y \in U^\tau : x \equiv_U y$
$\implies (x :: s) \leq_S (y :: t)$

SeqLeStabAppd0 $\forall s, t_1 \in (U^*)^\tau : s \leq_S t_1$
$\implies \forall t_0 \in (U^*)^\tau : s \leq_S (t_0 +\!+ t_1)$

SeqLeStabAppd1 $\forall s, t_0 \in (U^*)^\tau : s \leq_S t_0$
$\implies \forall t_1 \in (U^*)^\tau : s \leq_S (t_0 +\!+ t_1)$

SeqLeStabAppd $\forall s_0, s_1, t_0, t_1 \in (U^*)^\tau : s_0 \leq_S t_0 \implies s_1 \leq_S t_1$
$\implies (s_0 +\!+ s_1) \leq_S (t_0 +\!+ t_1)$

SeqLeRefl $\forall s, t \in (U^*)^\tau : t \in U^* \implies s \equiv_S t \implies s \leq_S t$

SeqLeTrans $\forall s, t, u \in (U^*)^\tau : u \in U^* \implies s \leq_S t \implies t \leq_S u \implies s \leq_S u$

SeqLePreorder $\leq_S$ is a preorder on U^*

FinSeqDirSet $(U^*, \leq_S)$ is a directed set (with upper bounds given by ++)

1.2.11 Finite Sequences by Repetition

Next, we define a way to construct finite sequences by repeating an element. Given $x \in U$ and $n \in \mathbb{N}$, we define the finite sequence $[x]_n$ as follows:

$$[x]_0 = [] \qquad\qquad [x]_{n+1} = x :: [x]_n$$

1.2.12 Properties of Finite Sequences by Repetition

We can now prove some properties of the $[x]_n$ construction.

SeqRepLength $\forall x \in U^\tau : \forall n \in \mathbb{N} : |[x]_n| = n$

SeqRepElem $\forall x \in U^\tau : \forall n \in \mathbb{N} : x \in_S [x]_{n+1}$

SeqRepSub $\forall x \in U^\tau : \forall m, n \in \mathbb{N} : [x]_m \subseteq_S [x]_n$

SeqRepInToSub $\forall x \in U^\tau : \forall s \in (U^*)^\tau : x \in_S s \implies \forall n \in \mathbb{N} : [x]_n \subseteq_S s$

SeqRepLe $\forall x \in U^\tau : \forall m, n \in \mathbb{N} : m \leq n \implies [x]_m \leq_S [x]_n$

SeqRepEq $\forall x, y \in U^\tau : \forall n, m \in \mathbb{N} : x \equiv_U y \implies n = m \implies [x]_n \equiv_S [y]_m$

SeqRepFinSeq $\forall x \in U : \forall n \in \mathbb{N} : [x]_n \in U^*$

1.2.13 Extending Finite Sequences

Finally, we define a way to extend finite sequences with elements already in the sequence. Given $s \in U^*$ and $n \in \mathbb{N}$, we define the finite sequence s^{+n} as follows:

$$[]^{+n} = [] \qquad\qquad (x :: s)^{+n} = [x]_n +\!\!+ (x :: s)$$

1.2.14 Properties of Extending Finite Sequences

We can now prove some properties of the s^{+n} construction.

SeqExtZero $\forall s \in (U^*)^\tau : s^{+0} = s$

SeqExtLen $\forall x \in U^\tau : \forall s \in (U^*)^\tau : \forall n \in \mathbb{N} : |(x :: s)^{+n}| = |x :: s| + n$

SeqExtLenEqToNil $\forall s \in (U^*)^\tau : |s^{+n}| = |s| \implies 0 < n \implies s = []$

SeqExtSub $\forall s \in (U^*)^\tau : \forall n \in \mathbb{N} : s \subseteq_S s^{+n}$

SeqExtLe $\forall s \in (U^*)^\tau : \forall n \in \mathbb{N} : s \leq_S s^{+n}$

SeqExtCompat $\forall s, t \in (U^*)^\tau : t \in U^* \implies s \equiv_S t$
$\implies \forall n \in \mathbb{N} : s^{+n} \equiv_S t^{+n}$

SeqSubCompatExt $\forall s, t \in (U^*)^\tau : t \in U^* \implies s \subseteq_S t$
$\implies \forall n \in \mathbb{N} : s^{+n} \subseteq_S t^{+n}$

SeqLeCompatExt $\forall s, t \in (U^*)^\tau : t \in U^* \implies s \leq_S t$
$\implies \forall n \in \mathbb{N} : s^{+n} \leq_S t^{+n}$

SeqExtFinSeq $\forall s \in U^* : \forall n \in \mathbb{N} : s^{+n} \in U^*$

2 Uniform Spaces

Now, that we have defined finite sequences in setoids, we can use them to define uniform spaces. We first define pseudometrics, which we then use to define uniform spaces.

2.1 Pseudometrics

2.1.1 Definition

A pseudometric on a setoid $(U, \equiv_U)$ is a map $d : U \times U \to \mathbb{R}^\tau$ satisfying:

$$\forall x, y \in U : x \equiv_U y \implies d(x, y) = 0 \qquad \text{(Indiscernibility)}$$
$$\forall x, y \in U : d(x, y) = d(y, x) \qquad \text{(Symmetry)}$$
$$\forall x, y, z \in U : d(x, z) \leq d(x, y) + d(y, z) \qquad \text{(Triangle Inequality)}$$

2.1.2 Properties

We can now prove some properties of pseudometrics.

PsMtrReal $\forall x, y \in U : d(x, y) \in \mathbb{R}$ (satisfies the "real" predicate)

PsMtrNNeg $\forall x, y \in U : 0 \leq d(x, y)$

PsMtrCompatLeft $\forall x, y, z \in U : x \equiv_U y \implies d(x, z) = d(y, z)$

PsMtrCompatRight $\forall x, y, z \in U : y \equiv_U z \implies d(x, y) = d(x, z)$

PsMtrCompat $\forall x, y, z, w \in U : x \equiv_U y \wedge z \equiv_U w \implies d(x, z) = d(y, w)$

PsMtrUB $\forall x, y, z \in U : d(x, z) - d(y, z) \leq d(x, y)$

PsMtrUB2 $\forall x, y, z \in U : d(x, y) - d(x, z) \leq d(y, z)$

2.2 Uniform Spaces

We can now define uniform spaces using pseudometrics.

2.2.1 Uniform Space

A uniform space is a pair (U, I, D) where $(U, \equiv_U)$ and $(I, \leq_I)$ are setoids, $D : I \times U \times U \to \mathbb{R}^\tau$ is a map assigning to each element i of I a pseudometric d_i on U, and the following properties hold:

$$\forall x, y \in U : \forall i \in I : d_i(x, y) = 0 \implies x \equiv_U y$$
$$\forall x, y \in U : \forall i, j \in I : i \equiv_I j \implies d_i(x, y) = d_j(x, y)$$

2.2.2 Maximum Metric

Given a uniform space (U, I, D), we can define the maximum metric $d_{\max}$: $U \times U \times I^* \to \mathbb{R}^\tau$ as follows:

$$d_{\max}(x, y, []) = 0 \qquad d_{\max}(x, y, i :: s) = \max(d_i(x, y), d_{\max}(x, y, s))$$

2.2.3 Properties of the Maximum Metric

We can now prove some properties of the $d_{\max}$ construction.

MaxMtrReal $\forall x, y \in U : \forall s \in I^* : d_{\max}(x, y, s) \in \mathbb{R}$

MaxMtrZero $\forall x, y \in U : \forall s \in I^* : x \equiv_U y \implies d_{\max}(x, y, s) = 0$

MaxMtrSymm $\forall x, y \in U : \forall s \in I^* : d_{\max}(x, y, s) = d_{\max}(y, x, s)$

MaxMtrTriIneq $\forall x, y, z \in U : \forall s \in I^* :$
$d_{\max}(x, z, s) \leq d_{\max}(x, y, s) + d_{\max}(y, z, s)$

MaxMtrPSMtr $\forall s \in I^* : d_{\max}(\cdot, \cdot, s)$ is a pseudometric on U

2.2.4 Class of Nets

Next, we want to introduce nets in uniform spaces, which are the analogue of sequences in metric spaces. Later on, we will then enforce a certain equality on these nets and will thus have to move to the equivalence classes of these nets. To this end, we will already begin introducing the class of nets here. Given a uniform space (U, I, D) and a directed set $(S, \leq_S)$, a map $f : S \to U$ is a net-class on U over S, if it satisfies:

$$\forall s \in S : f(s) \in U$$

Note, that we do not enforce compatibility with the equivalence relation on U.

2.2.5 Nets

A net on a uniform space (U, I, D) over a directed set $(S, \leq_S)$ is then a net-class $f : S \to U$ that is compatible with the equivalence relation on U, i.e., it satisfies:

$$\forall s, t \in S : s \equiv_S t \implies f(s) \equiv_U f(t)$$

Thus, nets are simply functions between setoids.

2.2.6 Convergence of Classes of Nets

We can now define the notion of convergence for classes of nets. Given a uniform space (U, I, D), a directed set $(S, \leq_S)$, and a net-class $f : S \to U$, we say that f converges to some $x \in U$ with modulus of convergence $m : I^* \to S$, if m is a setoid function, and the following holds:

$$\forall i \in I^* : \forall s \in S : m(i) \leq_S s \implies d_{\max}(f(s), x, i) \leq 2^{-|i|}$$

2.2.7 Uniqueness of Limits of Net-Classes

We now want to show that limits of converging net-classes are unique. To this end, we first prove:

NetClsConvUniqAux0 $\forall x \in \mathbb{R} : 0 \leq x \implies (\forall p \in \mathbb{N}^+ : x \leq 2^{-p}) \implies x = 0$

MaxMtrRepEq $\forall x, y \in U : \forall s \in I^* : \forall n \in \mathbb{N}^+ :$ $d_{\max}(x, y, s) = d_{\max}(y, x, s^{+(n+1)})$

Using these, we can now show:

NetClsConvUniq Given a uniform space (U, I, D), a directed set $(S, \leq_S)$, and a net-class $f : S \to U$ that converges to both $x, y \in U$ with moduli of convergence $m_x, m_y : I^* \to S$, then $x \equiv_U y$.

2.2.8 Convergence of Nets

Convergence of nets is defined simply as convergence of the underlying net-class. Thus, uniqueness of limits also holds for nets.

2.2.9 Cauchy Net-Classes

Next, we define the notion of Cauchy net-classes. Given a uniform space (U, I, D), a directed set $(S, \leq_S)$, and a net-class $f : S \to U$, we say that f is a Cauchy net-class with modulus $m : I^* \to S$, if m is a setoid function, and the following holds:

$$\forall i \in I^* : \forall s, t \in S : m(i) \leq_S s \wedge m(i) \leq_S t \implies d_{\max}(f(s), f(t), i) \leq 2^{-|i|}$$

2.2.10 Cauchy Nets

Cauchy nets are defined as Nets whose underlying net-class is Cauchy.

2.2.11 Regular Net-Classes

Our endgoal is to construct completions of uniform spaces. This will be similiar to the case of metric spaces, where we constructed the completion by going to the set of cauchy sequences. However, in uniform spaces, we have to be a bit more careful, as net(classes) also depend on both a modulus and a directed set. Thus, we will restrict ourselves to a special kind of Cauchy net-classes, called regular net-classes. Given a uniform space (U, I, D), a regular net-class is a Cauchy net-class $f : (I^*, \equiv_S, \leq_S) \to U$ over the directed set of finite sequences in I, with modulus $m(s) := s$. Furthermore, we require that f satisfies:

$$\forall i, j \in I : i \equiv_I j \implies \left(d_i(f([i]_n), f([j]_n)) \xrightarrow{n\to\infty} 0\right)$$

This is a strictly weaker condition than requiring that f is a net, as it only requires compatibility in the limit. Notably, if the underlying net-class of a regular net-class is a net, then this condition is automatically satisfied. We choose this definition, as it still holds for the equivalence classes of regular net-classes that we will later construct, while being enough to show compatibility of the pseudometrics on the completion. We write $\tilde{U}$ for the set (actually setoid) of regular net-classes on U.

2.2.12 Regular Nets

Given a uniform space (U, I, D), a regular net f is a cauchy net on $(I^*, \equiv_S, \leq_S)$ with modulus $m(s) := s$. We can then show that f is also a regular net-class. We write $\hat{U}$ for the set (actually setoid) of regular nets on U.

2.2.13 A Pseudometric on Regular Net-Classes

We now want to define a pseudometric on the set of regular net-classes. Given a uniform space (U, I, D), we define the map $D_{reg} : I \times \tilde{U} \times \tilde{U} \to \mathbb{R}^\tau$ as follows:

$$D_{reg}(i, F, G) = \lim_{n \in \mathbb{N}} d_i(F([i]_n), G([i]_n))$$

2.2.14 The regular metric converges

The first goal is to show that the limit in the definition of D_{reg} actually converges, i.e. that it is a real number. To this end, we first prove:

RegMtrRealAux0 $\forall F \in \tilde{U} : \forall i \in I : \forall p \in \mathbb{N}^+ : \forall m, n \in \mathbb{N} :$
$p + 1 \leq m \wedge p + 1 \leq n \implies d_{\max}(F([i]_m), F([i]_n), [i]_{p+1}) \leq 2^{-(p+1)}$

RegMtrRealAux1 $\forall F, G \in \tilde{U} : \forall i \in I : \forall m, n \in \mathbb{N} :$
$d_i(F([i]_m), G([i]_n)) \in \mathbb{R}$

Using this, we can prove:

RegMtrClsCauchy Let $F, G \in \tilde{U}$ and $i \in I$. Then the sequence

$$d_i(F([i]_n), G([i]_n))$$

is a Cauchy sequence with Cauchy modulus $m(p) := p + 1$.

Using the completeness of the real numbers, we obtain:

RegMtrClsReal $\forall F, G \in \tilde{U} : \forall i \in I : D_{reg}(i, F, G) \in \mathbb{R}$

Wir erhalten trivialerweise analoge Eigenschaften für reguläre Netze.

2.2.15 Properties of the Regular Pseudometric

We can now prove some more properties of the D_{reg} construction.

RegMtrClsEqdZero $\forall F \in \tilde{U} : \forall i \in I : D_{reg}(i, F, F) = 0$

RegMtrClsSymm $\forall F, G \in \tilde{U} : \forall i \in I : D_{reg}(i, F, G) = D_{reg}(i, G, F)$

RegMtrClsTriIneq $\forall F, G, H \in \tilde{U} : \forall i \in I :$
$D_{reg}(i, F, H) \leq D_{reg}(i, F, G) + D_{reg}(i, G, H)$

RegMtrClsNNeg $\forall F, G \in \tilde{U} : \forall i \in I : 0 \leq D_{reg}(i, F, G)$

Note that we cannot show that D_{reg} is a pseudometric, since the underlying space $\tilde{U}$ is not yet known to be a setoid. This will only be shown after we have introduced an equivalence relation on $\tilde{U}$.

2.2.16 Equivalence of Regular Net-Classes

We can now define an equivalence relation on the set of regular net-classes. Given a uniform space (U, I, D), we define the relation $\equiv_{\tilde{U}}$ on $\tilde{U}$ as follows:

$$\forall F, G \in \tilde{U} : F \equiv_{\tilde{U}} G \iff \forall i \in I : D_{reg}(i, F, G) = 0$$

Note, that under this equivalence relation, the "element-of" operation is not well-defined anymore. Thus, it makes no sense to speak about regular nets under this equivalence relation. This also shows why we had to introduce regular net-classes instead of just regular nets, since the properties of regular net-classes are stable under this equivalence relation.

2.2.17 Properties of the Equivalence of Regular Net-Classes

We can now prove some properties of the $\equiv_{\tilde{U}}$ relation.

RegNetEqRefl $\forall F \in \tilde{U} : F \equiv_{\tilde{U}} F$

RegNetEqSymm $\forall F, G \in \tilde{U} : F \equiv_{\tilde{U}} G \implies G \equiv_{\tilde{U}} F$

RegNetEqTrans $\forall F, G, H \in \tilde{U} : F \equiv_{\tilde{U}} G \implies G \equiv_{\tilde{U}} H \implies F \equiv_{\tilde{U}} H$

RegNetEqvRel $\equiv_{\tilde{U}}$ is an equivalence relation on $\tilde{U}$

RegNetClsEqSet $(\tilde{U}, \equiv_{\tilde{U}})$ is a setoid

Since we now have an equivalence relation on $\tilde{U}$, we can finally show that D_{reg} is a pseudometric on the setoid $(\tilde{U}, \equiv_{\tilde{U}})$.

2.2.18 The Uniform Space of Regular Net-Classes

We now want to show that the triple $(\tilde{U}, I, D_{reg})$ is a uniform space. We begin by showing that D_{reg} is actually a family of pseudometrics on the setoid $(\tilde{U}, \equiv_{\tilde{U}})$.

RegMtrZero $\forall F, G \in \tilde{U} : F \equiv_{\tilde{U}} G \implies \forall i \in I : D_{reg}(i, F, G) = 0$

RegMtrPsMtr $\forall i \in I : D_{reg}(\cdot, \cdot, i)$ is a pseudometric on the setoid $(\tilde{U}, \equiv_{\tilde{U}})$

We now have one remaining property to show in order to conclude that $(\tilde{U}, I, D_{reg})$ is a uniform space. Namely, that D_{reg} is compatible with the equivalence relation on I. This is by far the most difficult property to show, as it requires a lot of manipulations with limits and the triangle inequality, which are difficult to carry out in Minlog. Informally, the proof goes as follows:

$$\begin{aligned}
&\forall i, j \in I : \forall F, G \in \tilde{U} : i \equiv_I j \implies \\
&\quad d_i(F([i]_n), G([i]_n)) - d_j(F([j]_n), G([j]_n)) \\
&\qquad = d_i(F([i]_n), G([i]_n)) - d_i(F([j]_n), G([j]_n)) \\
&\qquad \leq d_i(F([i]_n), F([j]_n)) + d_i(F([j]_n), G([j]_n)) - d_i(F([j]_n), G([j]_n)) \\
&\qquad \leq \underbrace{d_i(F([i]_n), F([j]_n))}_{\to 0} + \underbrace{d_i(G([i]_n), G([j]_n))}_{\to 0} \to 0
\end{aligned}$$

In Minlog, we first prove the following basic results about limits:

RealCauchyEq Let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be pointwise equal sequences of real numbers. Then (a_n) is a Cauchy sequence with modulus m if and only if (b_n) is a Cauchy sequence with modulus m.

RealLimEq Let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be pointwise equal sequences of real numbers, where (a_n) is Cauchy. Then they have the same limit.

RealLimLe Let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be sequences of real numbers, where (a_n) converges to a and (b_n) converges to b. If $\forall n \in \mathbb{N} : a_n \leq b_n$, then $a \leq b$.

RealCauchyInv Let $(a_n)_{n\in\mathbb{N}}$ be a Cauchy sequence of real numbers. Then the sequence $(-a_n)_{n\in\mathbb{N}}$ is also a Cauchy sequence (with the same modulus).

RealLimInv Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of real numbers converging to a. Then the sequence $(-a_n)_{n\in\mathbb{N}}$ converges to $-a$.

RealCauchyPlus Let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be Cauchy sequences of real numbers with moduli m_a and m_b. Then the sequence $(a_n + b_n)_{n\in\mathbb{N}}$ is also a Cauchy sequence with modulus $m(p) := \max(m_a(p+1), m_b(p+1))$.

RealCauchyPlusEqMod Let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be Cauchy sequences of real numbers with the same modulus m. Then the sequence $(a_n + b_n)_{n\in\mathbb{N}}$ is also a Cauchy sequence with modulus $p \mapsto m(p+1)$.

RealLimPlus Let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be sequences of real numbers converging to a and b. Then the sequence $(a_n + b_n)_{n\in\mathbb{N}}$ converges to $a + b$.

Using these, we can now show the following auxiliary results:

RegMtrClsIdxEqAux0 Let $F \in \tilde{U}$ and $i, j \in I$ with $i \equiv_I j$. Then the sequence

$$d_i(F([i]_n), F([j]_n))$$

is a Cauchy sequence with Cauchy modulus $m(p) := p + 1$.

RegMtrClsIdxEqAux1 Let $F \in \tilde{U}$ and $i, j \in I$ with $i \equiv_I j$. Then

$$\lim_{n \in \mathbb{N}} d_i(F([i]_n), F([j]_n)) = 0$$

We can now finally show:

RegMtrIdxEq

$$\begin{aligned} &\forall i, j \in I : i \equiv_I j \implies \\ &\quad \forall F, G \in \tilde{U} : D_{reg}(i, F, G) = D_{reg}(j, F, G) \end{aligned}$$

Thus, we have shown all properties required to conclude:

RegNetClsUnfmSpace $(\tilde{U}, I, D_{reg})$ is a uniform space

2.2.19 Inclusion of the Original Uniform Space

Finally, we want to show that the original uniform space (U, I, D) is included in the uniform space of regular net-classes $(\tilde{U}, I, D_{reg})$. To this end, we define the map $\iota : U \to \tilde{U}$ as follows:

$$\forall x \in U : \iota(x)(s) = x$$

2.2.20 Properties of the Inclusion

We can now prove some properties of the ι construction.

InclRegNet $\forall x \in U : \iota(x) \in \hat{U}$

InclRegNetCls $\forall x \in U : \iota(x) \in \tilde{U}$

MaxMtrRegMtrIncl $\forall x, y \in U : \forall s \in I^* : d^U_{\max}(x, y, s) = D^{\tilde{U}}_{max}(\iota(x), \iota(y), s)$

MaxMtrRegMtrInclUBAux0 $\forall x, y \in U : \forall s \in I^* : \forall r \in \mathbb{R} :$
$0 \leq r \implies \forall i \in I : i \in_S s \implies d_i(x, y) \leq r \implies d^U_{\max}(x, y, s) \leq r$

MaxMtrRegMtrInclUBAux1 $\forall x, y \in U : \forall s \in I^* : \forall i \in I :$
$i \in_S s \implies d_i(x, y) \leq d^U_{\max}(x, y, s)$

MaxMtrRegMtrInclUBAux2 Let $F \in \tilde{U}$, $x \in U$, and $i \in I$. Then the sequence

$$d_i(F([i]_n), x)$$

is a Cauchy sequence with Cauchy modulus $m(p) := p + 1$.

MaxMtrRegMtrInclUB $\forall F \in \tilde{U} : \forall s, t \in I^* : s \leq_S t \implies$
$d^{\tilde{U}}_{\max}(F, \iota(F(t)), s) \leq 2^{-|t|}$

2.2.21 Complete Uniform Space

We now get to complete unform spaces. A uniform space (U, I, D) is called complete, if every Cauchy net on U converges to some limit in U.

2.2.22 The Completion of a Uniform Space

We now want to show that $(\tilde{U}, I, D_{reg})$ is the completion of (U, I, D). To this end, we first prove:

NetLimNetCls Let $(S, \leq_S)$ be a directed set, and $\tilde{f} : S \to \tilde{U}$ be a cauchy net on $(\tilde{U}, I, D_{reg})$ with module m. Then the map $f \in \left(\tilde{U}\right)^{\tau}$ defined as

$$\forall s \in S : f(s) = \tilde{f}(m(s^{+2}))(s^{+2})$$

is a cauchy net-class on U with modulus m.

2.2.23 Work left to do

This is where we are currently at in terms of formalization. The remaining steps to show that $(\tilde{U}, I, D_{reg})$ is the completion of (U, I, D) are as follows:

Show that the net-class f defined in NetLimNetCls is a regular net-class.

Show that $\tilde{f}$ converges to f in $(\tilde{U}, I, D_{reg})$.

The hardest part will likely be to show the first step, as this will again require a lot of manipulations with limits and the triangle inequality. The second step should then be relatively straightforward once the first step is done.