Convergence of series in real numbers

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This work treats the convergence of series as it is embedded in the file "rseq.scm" . After some recap of the convergence in $\mathbb{R}$ and the definition of Series, this work will cover some criteria for the convergence of series, namely the comparison test as well as the ratio test. After that the Cauchy product for absolute convergent series shall be proven. As said, all of the things done are also realized in minlog in the file "rseq.scm" and it will be frequently referred to the Lemmas and the proof in minlog. The work follows in some parts the Chapters 2.3 and 2.6 of the lecture script of "Constructive analysis with witnesses" by Helmut Schwichtenberg in WiSe 23/24.

1 Convergence of sequences in $\mathbb{R}$

On $\mathbb{R}$ all the Cauchy and Convergence predicates are formulated in two ways in Minlog, one as *RConv...* and one as *RealConv....* The former one stands just for the basic conditions of the convergence, the ladder demands for the modulus to be monotone and the sequence to be Real in every element.

Definition 1.1 ("RealCauchy") A sequence xs in $\mathbb{R}$ is a Cauchy sequence with a modulus $M : \mathbb{P} \to \mathbb{N}$, if and only if $\forall p \in \mathbb{P}$ and $\forall n, m \geq M(p)$ it holds:

$$|xs_n - xs_m| \leq 2^{-p}$$

In minlog this is written as $RCauchy\ xs\ M$ resp. $RealCauchy\ xs\ M$.

The limit ("RealLim") of a sequence of real numbers in minlog is defined by a programm constant:

$$(add-program-constant\ "RealLim"\ (py"(\mathbb{N} => \mathbb{R}) => (\mathbb{P} => \mathbb{N}) => \mathbb{R}"))$$
$$(add-computation-rules$$
$$"RealLim\ xs\ M"$$
$$"RealConstr([n](xs\ n)seq((xs\ n)mod(cNatPos\ n)))$$
$$([p]M(PosS\ p)max\ PosS(PosS\ p))")$$

or in its deanimated Form: $cRLim$.

In the Lemma "*RealLimReal*" it is shown, that assuming xs is a Cauchy sequence with modulus M, then $cRLim\ xs\ M$ is again a real number. Also the Completeness of the real numbers have already been shown in this Seminar, so will be here left out.

Definition 1.2 ("RealConvLim") A sequence in the real numbers xs is called to converge to a $x \in \mathbb{R}$ with modulus M, if $\forall p \in \mathbb{P}$ and $\forall n \geq M(p)$ it holds:

$$|xs_n - x| \leq 2^{-p}$$

In minog this is written as $RConvLim\ xs\ x\ M$ resp. $RealConvLim\ xs\ x\ M$.

There are many characteristics of this predicate proven in minlog, mainly the uniqueness of the limit: "RealConvLimUniq" and that every Sequence converging to a real number is also a Cauchy sequence: "RealConvLimToCauchy".

There are especially also the following limit theorems on two sequences xs, ys with the limits $x, y \in \mathbb{R}$ in with the moduli: $xs \xrightarrow{\text{M}} x$ and $ys \xrightarrow{\text{N}} y$, from which the last one will be useful in the proof of the Cauchy product:

1. RealConvLimLe: $\forall n \in \mathbb{N} : xs_n \leq ys_n \implies x \leq y$
2. RealConvLimPlus: $(xs_n + ys_n) \xrightarrow{\text{[p] max\{M(p+1),N(p+1)\}}} x + y$
3. RealConvLimUMinus: $(-xs_n) \xrightarrow{\text{M}} -x$
4. RealConvLimTimes: If furthermore $\forall n \in \mathbb{N} : |xs_n| \leq 2^p$ and $|y| \leq 2^q$ it follows:

$$(xs_n \cdot ys_n)_n \xrightarrow{\text{[r] max\{M(q+r+1),N(p+r+1)\}}} x \cdot y$$

2 Convergence of Series

In this chapter the main notions of the convergence of series are refreshed.

Definition 2.1 Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of reals, and define

$$s_n := \sum_{m=0}^{n} x_m.$$

We call s_n a *partial sum* of the sequence (x_n). Recall that in minlog the sums of real numbers are defined by giving the sequence xs the starting point 0 as well as the number of summands n: $(RealSum\ xs\ 0\ n)$.

The sequence

$$(s_n)_{n\in\mathbb{N}} = \left(\sum_{m=0}^{n} x_m\right)_{n\in\mathbb{N}} =: \sum_{m=0}^{\infty} x_m$$

is called the *series* determined by the sequence $(x_n)_{n\in\mathbb{N}}$.

We say that the series $\sum_{m=0}^{\infty} x_m$ *converges* if and only if the sequence (s_n) converges. For the convergence of series in minlog there exists the predicate ($RealSerConv$, resp. $RSerConv$) dependent on a real sequence xs together with a modulus of the convergence M, that in this case is required to be monotone.

$$(add - ids\ (list\ (list\ "RSerConv"\ (make - arity\ (py\ "\mathbb{N} => \mathbb{R}")\ (py\ "\mathbb{P} => \mathbb{N}"))))$$
$$("\ \forall\ xs, M(\forall p, n, m(Mp <= n\ \rightarrow\ abs(RealSum\ n\ m\ xs) <<= (1\#2^{**}p)) \rightarrow$$
$$RSerConv\ xs\ M)"\ "RSerConvIntro"))$$

It is shown in minlog ("RealSerConvLimToRealConvLimSum"), that the convergence of a series with this predicate coincides with the convergence of the sequence formed by the partial sums $([n]\ RealSum\ xs\ 0\ n)$.

In a previous speech we have also already treated the Cauchy criterion, which follows directly from the definition of the predicate:

A series $\sum_{m=0}^{\infty} x_n$ converges if and only if for all $p \in \mathbb{Z}$ there exists a $N \in \mathbb{N}$ such that for all $n, m \in \mathbb{N}$:

$$n \geq m \geq N \implies \left|\sum_{\nu=m}^{n} x_\nu\right| \leq \frac{1}{2^p}.$$

Definition 2.2 A Series is called absolutely convergent, if:

$$\sum_{m=0}^{\infty} |x_n|$$

In minlog there is no extra predicate for the absolute convergence, it is rather referred as:

$$RealSerConv([n]abs(xs\ n))\ M.$$

In the Lemma "$RealSerAbsConvToConv$" there is shown, that absolute convergence implies the convergence in its original sense.

3 Criteria of convergence

By the following theorem, we can show the convergence of Series by comparing each sequence member to a bigger one of another convergent series.

Theorem 3.1 *("RealComparisonTestMax") Let $\sum_{n=0}^{\infty} y_n$ be a convergent series and and x_n another sequence, with $|x_n| \leq y_n$ for all $n \geq m$ for a $m \in \mathbb{N}$. The sequence y_n is ≥ 0 for all $n \in \mathbb{N}$. It follows, that $\sum_{n=0}^{\infty} x_n$ is absolutely convergent.*

Proof. To show that $\sum_{n=0}^{\infty} |x_n|$ converges, take a $p \in \mathbb{P}$. Since $\sum_{n=0}^{\infty} y_n$ converges, we have by the Cauchy criterion a $N \in \mathbb{N}$ such that for all $l \geq n \geq max\{N, m\}$,

$$\sum_{j=n}^{l} y_j \leq \frac{1}{2^p}.$$

But then, since $n \geq m$, also the following holds:

$$\sum_{j=n}^{l} |x_j| \leq \sum_{j=n}^{l} y_j \leq \frac{1}{2^p}.$$

■

In minlog the Theorem gets firstly proofed is for a strictly non-negative real sequence xs and is then generalized for the absolute value of any sequence in $\mathbb{R}$, like in the theorem above. The first one, namely "$RComparisonTestMax$", is then used for the proof of the actual one: "$RealComparisonTestMax$".

We furthermore proof the Lemma ("RealSerConvTimesConstR")in minlog, that will allow us to pull constant, bounded Factors out of a series.

Also there are Lemmata to shift the indices of series up or down and still have a convergent series.

From the comparison of series to the geometric series, we get furthermore the following test:

Theorem 3.2 *("RealRatioTestZero") Let $(x_n)_{n \in \mathbb{N}}$ be a sequence of real numbers and*

$$|x_{n+1}| \leq q|x_n| \quad \text{for all } n \geq 0$$

with $0 \leq q < 1$. Then the series $\sum_{n=n_0}^{\infty} x_n$ is absolutely convergent.

Proof. By assumption, we have for all n the equation $|x_{n+1}| \leq q|x_n|$ and by induction it quickly follows:

$$|x_n| \leq q^n |x_0|,$$

The geometric series $\sum_{n=0}^{\infty} q^n$ converges (because $0 \leq q < 1$), hence also $\sum_{n=0}^{\infty} q^n |x_0|$. From the comparison test, we can conclude the absolute convergence of $\sum_{n=0}^{\infty} x_n$. ■

For the same proof in minlog ("RealRatioTestZero"), we first need the convergence of the geometric series, which is stated in the Theorem: "RealCauchyExpToRealSerConvExp", saying: If a real number $0 \leq x <_p 1$ and we have am modulus M for $RCauchy\ ([n]\ x^n)M$ it follows:

$$\sum_{j=0}^{\infty} x^n \text{ converges with module } q \mapsto M(PosS(q+p))$$

A modulus for (RCauchy ([n]x^n) M) is derived in the proof of "GeomSeqRealConvLim", which states that the geometric sequence converges and hence by earlier proofs is a Cauchy sequence.

4 Cauchy Product of series

Our goal in this final section is the Cauchy product theorem. In non constructive Analysis, this is the following Theorem:

Theorem 4.1 *If $\sum_{i=0}^{\infty} x_i$ and $\sum_{j=0}^{\infty} y_j$ are two absolutely convergent series, then the product of those can be written as:*

$$\left(\sum_{i=0}^{\infty} x_i\right) \cdot \left(\sum_{j=0}^{\infty} y_j\right) = \sum_{n=0}^{\infty} \left(\sum_{i+j=n} x_i \cdot y_j\right)$$

The Theorem we want to prove is the following:

Theorem 4.2 *("RealCauchyProdLim") We assume the following assumptions:*

- *Let xs and ys be sequences $\mathbb{N} \to \mathbb{R}$*

  Let $\sum_{i=0}^{\infty} xs_i \xrightarrow{M} x \in \mathbb{R}$ and $\sum_{j=0}^{\infty} ys_j \xrightarrow{N} y \in \mathbb{R}$

  For all $n \in \mathbb{N}$: $|\sum_{i=0}^{n} xs_i| \leq 2^p$ and $|\sum_{j=0}^{n} ys_j| \leq 2^q$

- *K(r) = max{N(p+r+1), M(q+r+1)}*
- *Let $xs0_n = \sum_{i=0}^{n} |xs_i|$ and $ys0_n = \sum_{j=0}^{n} |ys_j|$*

  The sequences $xs0$ and $ys0$ are Cauchy sequences with moduli M0 resp. N0

  For all $n \in \mathbb{N}$: $xs0_n \leq 2^{p0}$ and $ys0_n \leq 2^{q0}$

- *K0(r) = max{ N0(p0+r+1), M0(q0+r+1)}*

Then for all $n \geq max\{2 \cdot K0(r+1), K(r+1)\}$ it holds:

$$|\sum_{i+j<n} xs_i \cdot ys_i - x \cdot y| \leq 2^{-r}$$

The variables with the same name as in the Theorem "RealCauchyProdLim" shall have the same type in all the lemmas below, even if its left out at some points.

Lemma 4.3 *("RealSumMinusSquareMod") For a real valued double sum xss and* $m \leq n$ *we have:*

$$\sum_{i,j<n} xss_{i,j} - \sum_{i,j<m} xss_{i,j} = \sum_{i,j<n} xss_{i,j} \cdot \mathbb{1}_{\{m \leq max\{i,j\}\}} = \sum_{m \leq max\{i,j\}}^{i,j<n} xss_{i,j}$$

Lemma 4.4 *("RealConvLimZStar") Let* xs, x, M, ys, y, N, p, q *and* K *be like in the first part of the assumptions of Theorem "RealCauchyProdLim". I.e.* $\sum_{i=0}^{\infty} xs_i \xrightarrow{M} x$ *and* $\sum_{i=0}^{n} xs_i \leq 2^{-p}$ *for all natural n, vice versa for ys and K(r) = max{N(p+r+1), M(q+r+1)}. Then for all* $r \in \mathbb{P}$ *and* $n \geq K(r)$*:*

$$|\sum_{i,j<n} x_i \cdot y_j - x \cdot y| \leq 2^{-r}$$

Lemma 4.5 *("RealUpperTriangLimZeroAux") Let* xs, $xs0$, $M0$, $p0$, ys, $ys0$, $q0$, $N0$ *and* $K0$ *like in the second part of the assumptions of "RealCauchyProdLim". Then for all* $r \in \mathbb{P}$ *and* $n \geq K0(r)$*:*

$$|x0_n \cdot y0_n - x0_{K0(r)} \cdot y0_{K0(r)}| \leq 2^{-r}$$

Lemma 4.6 *("RealLeAbsMinusZStarZMinusPStar) For real valued sequences xs and ys we have:*

$$|\sum_{i,j<n} x_i \cdot y_j - \sum_{i+j<n} x_i \cdot y_j| \leq \sum_{i+j \geq n}^{i,j<n} |x_i| \cdot |y_j|$$

Lemma 4.7 *("RealUpperTriangleMinusSquare") For real valued sequences xs and ys, and* $m + m \leq n$ *we have:*

$$\sum_{n \leq i+j}^{i,j<n} |xs_i| \cdot |ys_j| \leq \sum_{m \leq max\{i,j\}}^{i,j<n} |xs_i| \cdot |ys_j|$$

An lastly the main Lemma to proof the Cauchy product formula, depending on 4 of the five upper Lemma:

Lemma 4.8 *("RealUpperTriangLimZero") Let* xs, $xs0$, $M0$, $p0$, ys, $ys0$, $q0$, $N0$ *and* $K0$ *like in the second part of the assumptions of "RealCauchyProdLim". Then for all* $r \in \mathbb{P}$ *and* $n \geq 2 \cdot K0(r)$*:*

$$|\sum_{i,j<n} x_i y_j - \sum_{i+j<n} x_i y_j| \leq 2^{-r}$$