1 Completion of uniform spaces

As a generalization of the completion of metric spaces, this work will focus on the completion of uniform spaces. The study is primarily based on the work "Yet Another Predicative Completion of a Uniform Space" by Hajime Ishihara (July 1, 2020), in which he constructively develops this completion on the axioms of Elementary Constructive Set Theory ($ECST$) along with the exponentiation axiom (Exp). The steps for the completion of uniform spaces are designed closely parallel to those for the completion of metric spaces. However, instead of sequences, a strictly more general structure of nets will be introduced. The space of nets on the underlying set D will then represent a completion. Unlike the completion of metric spaces, it will become evident that the completion of uniform spaces does not involve all Cauchy nets but rather a selected subset of regular nets.

Definition 1.1 A pseudo-metric d in a Set X is a mapping $d : X \times X \to \mathbb{R}$, which fulfills the following clauses for all $x, y, z \in X$:

1. $d(x, x) = 0$
2. $d(x, y) = d(y, x)$
3. $d(x, y) \leq d(x, z) + d(z, y)$

The pseudo-metrics defined here (also referred to as semi-metrics) do not differ in their clauses from the metrics on metric spaces previously defined in the seminar. However, it should be noted that in this case, equality in the underlying set X does not necessarily require that $d(x, y) = 0$ implies $x = y$. There will most likely be elements that are not equal yet have a distance of 0.

Definition 1.2 A uniform space is a pair (X, D), consisting of a set X together with a set $D = \{d_\mathrm{i} \mid i \in I\}$ of pseudo-metrics d_i indexed by an index set $I \neq \emptyset$, such that:

$$\forall i \in I\, (d_i(x, y) = 0) \implies x = y$$

for all $x, y \in X$.

Uniform spaces form a strict generalization of metric spaces. These are obtained for $I = \{*\}$.

For the construction of nets on a uniform space afterwards, one will need the following construction of finite sequences as a quasi ordered set.

Definition 1.3 For a set S, we write $S^* = \{\langle s_0, ..., s_\mathrm{n}\rangle \mid n \in \mathbb{N}, s_\mathrm{i} \in S\}$ for the set of finite sequences of S with the following notations:

1. $|\sigma|$ denotes the length of $\sigma \in S^*$;
2. ε denotes the empty sequence with $|\varepsilon| = 0$;
3. $\sigma(l)$ denotes the l-th element of $\sigma \in S^*$, where $l < |\sigma|$;
4. $s \in \sigma$ denotes that $s = \sigma(l)$ for some $l < |\sigma|$;
5. $\sigma * \tau$ denotes the concatenation of $\sigma \in S^*$ and $\tau \in S^*$;
6. s^n denotes the constant sequence $\langle s, \dots, s\rangle$ of length n.

Now we define a relation $\preceq_S$ on S^* by

$$\sigma \preceq_S \tau \iff |\sigma| \leq |\tau| \wedge \forall s \in S\,(s \in \sigma \to s \in \tau)$$

for each $\sigma, \tau \in S^*$.

For any $s_0 \in S$ and $n \in \mathbb{N}$, we write $\sigma +^n$ for the sequence $\sigma * s_0^n$; note that $\sigma +^n \preceq_S \tau +^n$ whenever $\sigma \preceq_S \tau$.

Lemma 1.4 *For any set $S \neq \emptyset$ the pair $(S^*, \preceq_S)$ is a directed preordered set, which means, that the relation $\preceq_S$ is transitive and reflexive.*

Proof. To show the directionality, for any $\sigma, \tau \in S^*$ it is obvious, that $\sigma \preceq_S \sigma * \tau$ and $\tau \preceq_S \sigma * \tau$, as their individual length is smaller than the length of the concatenation and all of the Elements of σ and τ do also appear in $\sigma * \tau$. The reflexivity follows, because the length and the objects of the same Element always coincide. For transitivity, assume $\sigma \preceq_S \tau, \tau \preceq_S \nu \in S^*$. As $|\tau|$ is larger than $|\sigma|$, so is $|\nu|$ and vice versa with the elements of σ ∎

One observes, that this structure is not actually a partially ordered set, as the anti symmetry doesn't hold in general. One can for example just rearrange a finite set σ.

Definition 1.5 Let (X, D) be a uniform space with $D = \{d_i \mid i \in I\}$, and for each $\sigma \in I^*$, let d_σ be a pseudo-metric on X given by

$$d_\sigma(x, y) = \max\{d_i(x, y) \mid i \in \sigma\}$$

for each $x, y \in X$; if $\sigma = \varnothing$, then let $d_\sigma(x, y) = 0$.

Definition 1.6 (Net) Let $(\Lambda, \preceq)$ be a directed, partially ordered set and (X, D) a uniform space. A net (or Moore-Smith sequence) on Λ in X is a mapping $\Lambda \to X, \lambda \mapsto x_\lambda$. Like sequences it can be denoted by $(x_\lambda)_{\lambda \in \Lambda}$.

The nets are intended to take the role of sequences in uniform spaces. They are, in particular, a strict generalization of sequences, as can be seen when considering $S = \{*\}$. The sequences on this specific set are determined precisely by their length, just as the quasi-order on it corresponds to the less-than relation on the natural numbers. As quasi-ordered sets, they are therefore equivalent to the natural numbers. Nets over this set thus correspond exactly to sequences $\mathbb{N} \to X$.

Definition 1.7 (convergence of nets and Cauchy-nets) Let $(\Lambda, \preceq)$ be a directed, partially ordered set and (X, D) a uniform space. A net $(x_\lambda)_{\lambda\in\Lambda}$ on Λ in X converges to a $x \in X$, in symbols: $x_\lambda \to x$, with a modulus $\beta : I^* \to \Lambda$, if for each $\sigma \in I^*$ and $\lambda \in \Lambda$:

$$\beta(\sigma) \ \preceq \ \lambda \implies d_\sigma(x_\lambda, x) \leq 2^{-|\sigma|}$$

A Cauchy-net $(x_\lambda)_{\lambda\in\Lambda}$ is a net on Λ in X together with a modulus $\alpha : I^* \to \Lambda$, which fulfills the following formula for each $\sigma \in I^*$ and $\mu, \nu \in \Lambda$:

$$\alpha(\sigma) \ \preceq \ \mu, \nu \implies d_\sigma(x_\mu, x_\nu) \leq 2^{-|\sigma|}$$

Lemma 1.8 *(Uniqueness of limits) Let (X, D) be a uniform space. If a net (x_λ) in X converges to elements x and y of X, then $x = y$.*

Proof. Let $D = \{d_i \mid i \in I\}$, and suppose that a net (x_λ) on $(\Lambda, \prec)$ converges to $x \in X$ and also to $y \in X$ with the two moduli: $\alpha : I^* \to \Lambda$ and $\beta : I^* \to \Lambda$. By the directionality of $(\Lambda, \preceq)$ for each $i \in I$ and n, there exists $\lambda \in \Lambda$ such that $\alpha(i^n) \prec \lambda$ and $\beta(i^n) \prec \lambda$. Therefore the definition of the convergence provides:

$$d_i(x, y) = d_{i^n}(x, y) \leq d_{i^n}(x, x_\lambda) + d_{i^n}(x_\lambda, y) \leq 2^{-n} + 2^{-n} \leq 2^{n-1}.$$

As this holds for any $n \in \mathbb{N}$ the non negative distance $d_i(x, y)$ is already 0. This is also independent of the choice of $i \in I$ and hence from the definition of a uniform space $x = y$. ∎

Definition 1.9 (regular net) Let (X, D) be a uniform space and $D = \{d_i \mid i \in I\}$ the associated pseudo-metrics. A regular net in X is a Cauchy-net on $(I^*, \preceq_{I^*})$ with the modulus $id_{I^*} : I^* \to I^*$.

The space of all regular nets will be referred as $\tilde{X}$.

In the following section, it will be shown that for the completion of uniform spaces, it is sufficient to use the regular nets. This simplifies the proofs insofar as there is no need to carry an additional modulus for the Cauchy nets, since in regular nets, this is the identity on I^*.

For this space $\tilde{X}$ we want to define the structure of a uniform Space in the following way:

Definition 1.10 Let (X, D) be a uniform space, with $D = \{d_i \mid i \in I\}$. Then for each $i \in I$ we define a map: $\tilde{d}_i : \tilde{X} \times \tilde{X} \to \mathbb{R}$ by the following construction:

$$\forall x, y \in \tilde{X} : \tilde{d}_i(x, y) = \lim_{n\to\infty} d_i(x_{i^n}, y_{i^n})$$

This limit in $\mathbb{R}$ is well-defined and for each $i \in I$ $\tilde{d}_i$ is a pseudo-metric on $\tilde{X}$.

Proof. Let $x = (x_\rho), y = (y_\rho) \in \widetilde{X}$. Then we show that $(d_i(x_{i_n}, y_{i_n}))_n$ is a Cauchy sequence in $\mathbb{R}$ with a modulus $n \mapsto n+1$. In fact, for each $m, m' \geq n+1$, since

$$d_{i_{n+1}}(x_{i_m}, x_{i_{m'}}) \leq 2^{-(n+1)} \quad \text{and} \quad d_{i_{n+1}}(y_{i_m}, y_{i_{m'}}) \leq 2^{-(n+1)},$$

we have

$$\begin{aligned}
&d_i(x_{i_m}, y_{i_m}) - d_i(x_{i_{m'}}, y_{i_{m'}}) \\
&\leq d_i(x_{i_m}, x_{i_{m'}}) + d_i(x_{i_{m'}}, y_{i_{m'}}) + d_i(y_{i_{m'}}, y_{i_m}) - d_i(x_{i_{m'}}, y_{i_{m'}}) \\
&= d_{i_{n+1}}(x_{i_m}, x_{i_{m'}}) + d_{i_{n+1}}(y_{i_{m'}}, y_{i_m}) \\
&\leq 2^{-(n+1)} + 2^{-(n+1)} = 2^{-n}
\end{aligned}$$

We conclude, that $(d_i(x_{i_n}, y_{i_n}))_n$ is a Cauchy sequence, and hence it converges in $\mathbb{R}$.

The first two criteria of being a pseudo-metric are obvious, as for each $x, y \in \widetilde{X}$ $\widetilde{d}_i(x,x) = 0$ and $\widetilde{d}_i(x,y) = \widetilde{d}_i(y,x)$ follow from the same equalities on $d_i(x_{i_n}, y_{i_n})$. For the triangle inequality, we have

$$\begin{aligned}
\widetilde{d}_i(x,y) &= \lim_{n\to\infty} d_i(x_{i_n}, y_{i_n}) \\
&\leq \lim_{n\to\infty} d_i(x_{i_n}, z_{i_n}) + \lim_{n\to\infty} d_i(z_{i_n}, y_{i_n}) \\
&= \widetilde{d}_i(x,z) + \widetilde{d}_i(z,y)
\end{aligned}$$

for each $x, y, z \in \widetilde{X}$. ∎

Lemma 1.11 *Let (X, D) be a uniform space with $D = \{d_i \mid i \in I\}$. Define the inclusion map $\iota_X : X \to \widetilde{X}$ by*

$$(\iota_X(x))(\sigma) = x$$

for each $x \in X$ and $\sigma \in I^*$. Then

$$d_\sigma(x,y) = \widetilde{d}_\sigma(\iota_X(x), \iota_X(y))$$

for each $\sigma \in I^*$ and $x, y \in X$.

Proof. By plugging in the definition of the just defined pseudo-metric $\tilde{d}_i$, one gets for all $i \in I$ and $x, y \in X$:

$$\widetilde{d}_i(\iota_X(x), \iota_X(y)) = \lim_{n\to\infty} d_i(\iota_X(x)_{i^n}, \iota_X(y)_{i^n}) = \lim_{n\to\infty} d_i(x,y) = d_i(x,y).$$

Like so, this equation also holds for all $\sigma \in I^*$. ∎

Lemma 1.12 *Let (X, D) be a uniform space with $D = \{d_i \mid i \in I\}$, and let $x = (x_\rho) \in \tilde{X}$. Then*

$$\tilde{d}_\sigma(x, \iota_X(x_\tau)) \leq 2^{-|\tau|}$$

for each $\sigma, \tau \in I^*$ with $\sigma \preceq_I \tau$.

Proof. Assume $\sigma, \tau \in I^*$ with $\sigma \preceq_I \tau$ and choose $i \in \sigma$ and $n \in \mathbb{N}$. As x is regular we get for every $i^{|\tau|} \preceq_I \tau, \rho$ the inequality: $d_i(x_\rho, x_\tau) \leq 2^{-|\tau|}$. As from $i \in \sigma \preceq_I \tau$ it follows $i \in \tau$, one receives $i^{|\tau|} \preceq_I \tau$. The statement $|\tau| \leq n \in \mathbb{N} \implies i^{|\tau|} \preceq_I i^n$ follows instantly from definition. This leads to:

$$\forall n \geq |\tau| : d_i(x_{i^n}, x_\tau) \leq 2^{-|\tau|}$$

which implies:

$$\tilde{d}_i(x, \iota_X(x_\tau)) \leq 2^{-|\tau|}.$$

As this is independent of the choice of $i \in \sigma$, one reaches the desired: $\tilde{d}_\sigma(x, \iota_X(x_\tau)) \leq 2^{-|\tau|}$ ∎

Definition 1.13 (Completion of a uniform space) The completion of a uniform space (X, D) with $D = \{d_i \mid i \in I\}$ is the space $(X, \tilde{D})$ with $\tilde{D} = \{\tilde{d}_i \mid i \in I\}$, which becomes a uniform space with the equality $=_{\tilde{X}}$ given by

$$x =_{\tilde{X}} y \iff \forall i \in I\, (\tilde{d}_i(x, y) = 0)$$

for each $x, y \in \tilde{X}$.

We have already seen, that the embedding $\iota_X : X \to \tilde{X}$ is an isometry and its image lies dense in the Space $\tilde{X}$. To show that the above defined completion of the uniform space is really a completion in the former sense, it remains to show, that it is actually complete.

Theorem 1.14 *The completion $(\tilde{X}, \tilde{D})$ of a uniform space (X, D) is complete.*

Proof. Let $D = \{d_i \mid i \in I\}$, and suppose that $(x_\lambda)_{\lambda \in \Lambda} = ((x_{\lambda,\rho})_{\rho \in I^*})_{\lambda \in \Lambda}$ is a Cauchy net on $(\Lambda, \preceq)$ in $\tilde{X}$ with a modulus $\alpha : I^* \to \Lambda$. For each $\sigma \in I^*$, define a net $y = (y_\rho)$ on $(I^*, \preceq_I)$ in X by

$$y_\rho = x_{\alpha(\rho+2),\rho+2}$$

for each $\rho \in I^*$. We show that y is a regular net. Therefore consider $\sigma, \tau, \upsilon \in I^*$ with $\sigma \preceq_I \tau, \upsilon$. Then there exists $\lambda \in \Lambda$ such that $\alpha(\tau^{+2}) \preceq \lambda$ and $\alpha(\upsilon^{+2}) \preceq \lambda$, and, since $\sigma \preceq_I \tau^{+2}, \upsilon^{+2}$, we have

$$\begin{aligned}
d_\sigma(y_\tau, y_\upsilon) &= \tilde{d}_\sigma(\iota_X(x_{\alpha(\tau^{+2}),\tau^{+2}}), \iota_X(x_{\alpha(\upsilon^{+2}),\upsilon^{+2}})) \\
&\leq \tilde{d}_\sigma(\iota_X(x_{\alpha(\tau^{+2}),\tau^{+2}}), x_{\alpha(\tau^{+2})}) + \tilde{d}_\sigma(x_{\alpha(\tau^{+2})}, x_\lambda) + \tilde{d}_\sigma(x_\lambda, x_{\alpha(\upsilon^{+2})}) \\
&\quad + \tilde{d}_\sigma(x_{\alpha(\upsilon^{+2})}, \iota_X(x_{\alpha(\upsilon^{+2}),\upsilon^{+2}})) \\
&\leq 2^{-|\tau^{+2}|} + \tilde{d}_{\tau^{+2}}(x_{\alpha(\tau^{+2})}, x_\lambda) + \tilde{d}_{\upsilon^{+2}}(x_\lambda, x_{\alpha(\upsilon^{+2})}) + 2^{-|\upsilon^{+2}|} \\
&\leq 2^{-|\tau^{+2}|} + 2^{-|\tau^{+2}|} + 2^{-|\upsilon^{+2}|} + 2^{-|\upsilon^{+2}|} \\
&\leq 2^{-(|\tau|+1)} + 2^{-(|\upsilon|+1)} \\
&\leq 2^{-|\sigma|}.
\end{aligned}$$

using Lemma 1.12, the Cauchy property of x as well as the fact that $\tilde{d}_\sigma \leq \tilde{d}_\rho$ if $\sigma \preceq_I \rho$. Therefore y is regular. We further have to show, that $x_\lambda \to y$, hence we have to find a modulus of this convergence. We define $\beta : I^* \to \Lambda$ by

$$\beta(\sigma) = \alpha(\sigma^{+3})$$

for each $\sigma \in I^*$. If $\beta(\sigma) \preceq \lambda$, then

$$\begin{aligned}\tilde{d}_\sigma(x_\lambda, y) &\leq \tilde{d}_\sigma(x_\lambda, x_{\alpha(\sigma^{+3})}) + \tilde{d}_\sigma(x_{\alpha(\sigma^{+3})}, \iota_X(x_{\alpha(\sigma^{+3}),\sigma^{+3}})) + \tilde{d}_\sigma(\iota_X(y_{\sigma^{+1}}), y) \\ &\leq 2^{-|\sigma^{+3}|} + 2^{-|\sigma^{+3}|} + 2^{-|\sigma^{+1}|} < 2^{-|\sigma|},\end{aligned}$$

using the Cauchy property of x and two times Lemma 1.12. Therefore, (x_λ) converges to y with the modulus β. ■

The axioms used in Ishihara's work can all be implemented in Minlog within the theory of continuous functionals (TCF), which makes it easy to see that this completion of uniform spaces can indeed be embedded into Minlog. However, from the current standpoint, several components still need to be added beforehand.

Firstly, the definition of uniform spaces based on the type of pseudo-metrics is still missing.

Furthermore, the set of finite sequences of a set, together with the quasi-order defined on it, is not yet available as such. After that, one could incorporate the type of nets on a uniform space, then define convergence and the Cauchy property, and finally define regular nets as a specific type of Cauchy nets.

The theorems and lemmas are proven in great detail in the study of Hajime Ishihara and can also be implemented in Minlog in the manner described above.