Abstract Integration Spaces

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Def 1:

An abstract integration space (IS) is a pair (L, E) with:

(1) $(L, \vee)$ is a vector lattice

(2) $E : L \to \mathbb{R}$ is a positive, linear functional, i.e. $\forall x, y \in L, a \in \mathbb{R}$:

- (a) $E(ax + y) = aE(x) + E(y)$
- (b) $0 \leq x \Rightarrow 0 \leq E(x)$

Remark:

The axiom 2b is equivalent to:

$$\text{(b)} \iff \forall x, y \in L : x \leq y \Rightarrow E(x) \leq E(y)$$

Konvention:

FOr the following, we fix an integration space (L, E).

Example:

Consider the space of step functions on $\mathbb{R}$ $\mathcal{S}(\mathbb{R})$ with the usual integral $\int_{\mathbb{R}}$. Then $(\mathcal{S}(\mathbb{R}), \int_{\mathbb{R}})$ is an integration space.

Def 2:

We define the following maps:

(1) $\|\cdot\| : L \to \mathbb{R}$ with $\|x\| := E(|x|)$

(2) $d_s : L \times L \to \mathbb{R}$ with $d_s(x, y) := \|x - y\|$

Lemma 1:

The map $\|\cdot\|$ is a pseudonorm on L and d_s is a pseudometric on L.

Proof:

For all $x, y \in L$ and $a \in \mathbb{R}$ we have:

(1) $\|0\| = E(|0|) = E(0) = 0$

(2) $\|ax\| = E(|ax|) = E(|a||x|) = |a|E(|x|) = |a|\|x\|$

(3) $\|x+y\| = E(\underbrace{|x+y|}_{\leq |x|+|y|}) \leq E(|x|+|y|) = E(|x|) + E(|y|) = \|x\| + \|y\|$

Thus $\|\cdot\|$ is a pseudonorm, inducing the pseudometric d_s. $\square$

Lemma 2:

The following inequalities holds:

$$\forall x, y, x', y' \in L : \forall \circ \in \{+, \vee, \wedge\} : d_s(x \circ y, x' \circ y') \leq d_s(x, x') + d_s(y, y')$$

Proof:

Let $x, y, x', y' \in L$ be arbitrary.

(1) For $\circ = +$ we have:

$$\begin{aligned} d_s(x+y, x'+y') &= \|x+y-x'-y'\| = \|x-x'+y-y'\| \\ &\overset{\text{L1}}{\leq} \|x-x'\| + \|y-y'\| = d_s(x,x') + d_s(y,y') \end{aligned}$$

(2) For $\circ \in \{\vee, \wedge\}$ we have:

$$\begin{aligned} d_s(x \circ y, x' \circ y') &= \|x \circ y - x' \circ y'\| \\ &= \|x \circ y - x' \circ y + x' \circ y - x' \circ y'\| \\ \text{(L1)} &\leq \|x \circ y - x' \circ y\| + \|x' \circ y - x' \circ y'\| \\ &= E(\underbrace{|x \circ y - x' \circ y|}_{\text{(s.o.)}\leq |x-x'|}) + E(\underbrace{|x' \circ y - x' \circ y'|}_{\leq |y-y'|}) \\ \text{(Def. 1)} &\leq E(|x-x'|) + E(|y-y'|) = d_s(x,x') + d_s(y,y') \end{aligned}$$

$\square$

Def 3:

We turn (L, d_s) into a metric space by defining:

$$\forall x, y \in L : x =_L y :\iff d_s(x,y) = 0$$

Then let $(\mathcal{L}, \tilde{d}_s)$ be the metric space completion of (L, d_s). This induces the canonical embedding $\iota_{\mathcal{L}} : L \to \mathcal{L}$.

Lemma 3:

(1) The following maps are uniformly continuous:

$$(\iota_{\mathcal{L}} \circ +), (\iota_{\mathcal{L}} \circ \vee), (\iota_{\mathcal{L}} \circ \wedge) : L \times L \to \mathcal{L}$$

(2) The following map is locally uniformly continuous:

$$\iota_{\mathcal{L}} \circ (- \cdot -) : \mathbb{R} \times L \to \mathcal{L}$$

Proof:

(1) We consider (L, d_s) to be a uniform space (L, D_s) index by the singleton set: $D_s := \{d_i \mid i \in \{s\}\}$. Then our module must be of the form:

$$\alpha : \mathbb{N} \to (\{s\} + \{s\})^*$$

We denote the elements of $\{s\} + \{s\}$ as s_0 for the s of the left singleton and s_1 for the s of the right singleton. Then we define:

$$\alpha(n) := s_0^{n+1} * s_1^1$$

Now take $(x_0^i, x_1^i) \in L \times L$ for $i = 0, 1$. Then by definition we have:

$$d_{\alpha(n)}((x_0^0, x_1^0), (x_0^1, x_1^1)) = \max_{i=0,1} d_s(x_i^0, x_i^1) \quad \wedge \quad |\alpha(n)| = n + 2$$

Now fix a $n \in \mathbb{N}$ and the $(x_i^0, x_i^1) \in L \times L$ such that:

$$d_{\alpha(n)}((x_0^0, x_1^0), (x_0^1, x_1^1)) \leq 2^{-|\alpha(n)|}$$

Thus we get the following inequality:

$$\begin{aligned} \forall i = 0, 1 : d_s(x_i^0, x_i^1) \leq d_{\alpha(n)}((x_0^0, x_1^0), (x_0^1, x_1^1)) \leq 2^{-(n+2)} \\ \implies d_s(x_0^0, x_0^1) + d_s(x_1^0, x_1^1) \leq 2^{-(n+2)} + 2^{-(n+2)} < 2^{-n} \end{aligned}$$

Finally, by the above inequality, we get:

$$\begin{aligned} \forall \circ \in \{+, \vee, \wedge\} : \tilde{d}_s(\iota_{\mathfrak{L}}(x_0^0 \circ x_1^0), \iota_{\mathfrak{L}}(x_0^1 \circ x_1^1)) &= d_s(x_0^0 \circ x_1^0, x_0^1 \circ x_1^1) \\ &\overset{\text{L2}}{\leq} d_s(x_0^0, x_0^1) + d_s(x_1^0, x_1^1) < 2^{-n} \end{aligned}$$

Hence the maps are uniformly continuous with module α.

(2) We also consider $(\mathbb{R}, d_r)$ to be a uniform space index by $\{r\}$, where:

$$\forall a, b \in \mathbb{R} : d_r(a, b) := |a - b|$$

Now let ξ be a regular net in $(\mathbb{R}, d_r) \times (L, d_s)$ such that:

$$\xi = ((c_\sigma, z_\sigma))_{\sigma \in (\{r\}+\{s\})^*} \quad \wedge \quad \rho := r^1 * s^1 \in (\{r\} + \{s\})^*$$

Here we omit the subscripts r_0, s_0 for the sake of brevity. Now fix some $N \in \mathbb{N}$ such that $\max\{|c_\rho|, \|z_\rho\|\} \leq 2^N - 1$. Then we define:

$$\beta : \mathbb{N} \to (\{r\} + \{s\})^*, \qquad n \mapsto r^{N+n+1} * s^1$$

Now we fix some $n \in \mathbb{N}$ and $(a, x), (b, y) \in U_{\beta(n)}(\xi)$. By definition we have:

$$\forall (c, z) \in U_{\beta(n)}(\xi) : \tilde{d}_s(\iota_{\mathbb{R}\times L}(c, z), \xi) \leq 2^{-|\beta(n)|} = 2^{-(N+n+2)}$$

A first estimation yields $(*)$:

$$\begin{aligned} \tilde{d}_s(\iota_{\mathfrak{L}}(ax), \iota_{\mathfrak{L}}(by)) &= d_s(ax, by) = E(|ax - by|) \\ &\leq E(|ax - ay| + |ay - by|) = |a| d_s(x, y) + \|y\| d_r(a, b) \end{aligned}$$

We now individually estimate the parts of the above term:

(a) We have the following inequality:

$$\begin{aligned} d_{\beta(n)}((a, x), (b, y)) &\leq \tilde{d}_{\beta(n)}(\iota_{\mathbb{R}\times L}(a, x), \iota_{\mathbb{R}\times L}(b, y)) \\ &\leq \tilde{d}_{\beta(n)}(\iota_{\mathbb{R}\times L}(a, x), \xi) + \tilde{d}_{\beta(n)}(\xi, \iota_{\mathbb{R}\times L}(b, y)) \\ &\leq 2^{-(N+n+2)} + 2^{-(N+n+2)} = 2^{-(N+n+1)} \end{aligned}$$

$$\implies d_s(x, y), d_r(a, b) \leq d_{\beta(n)}((a, x), (b, y)) \leq 2^{-(N+n+1)}$$

(b) We have the following inequality:

$$\tilde{d}_\rho(\iota_{\mathbb{R}\times L}(c_\rho, z_\rho), \xi) + \tilde{d}_\rho(\xi, \iota_{\mathbb{R}\times L}(a, x)) \leq 2^{-|\rho|} + 2^{-|\rho|} = 2^{-2} + 2^{-2} < 1$$

Using this we get:

$$\begin{aligned} |c_\rho - a| &= \tilde{d}_{(0,r)}(\iota_{\mathbb{R}\times L}(c_\rho, z_\rho), \iota_{\mathbb{R}\times L}(a, x)) \\ &\leq \tilde{d}_{(0,r)}(\iota_{\mathbb{R}\times L}(c_\rho, z_\rho), \xi) + \tilde{d}_{(0,r)}(\xi, \iota_{\mathbb{R}\times L}(a, x)) \\ &\leq \tilde{d}_\rho(\iota_{\mathbb{R}\times L}(c_\rho, z_\rho), \xi) + \tilde{d}_\rho(\xi, \iota_{\mathbb{R}\times L}(a, x)) < 1 \\ \|z_\rho - y\| &= \tilde{d}_{(1,s)}(\iota_{\mathbb{R}\times L}(c_\rho, z_\rho), \iota_{\mathbb{R}\times L}(a, x)) \\ &\leq \tilde{d}_\rho(\iota_{\mathbb{R}\times L}(c_\rho, z_\rho), \xi) + \tilde{d}_\rho(\xi, \iota_{\mathbb{R}\times L}(a, x)) < 1 \end{aligned}$$

Thus we finally get:

$$\begin{aligned} |a| = |c_\rho + a - c_\rho| \leq |c_\rho - a| + |c_\rho| \leq 1 + 2^N - 1 = 2^N \\ \|y\| = \|z_\rho + y - z_\rho\| \leq \|z_\rho - y\| + \|z_\rho\| \leq 1 + 2^N - 1 = 2^N \end{aligned}$$

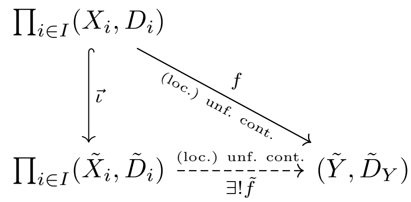
Substituting (a) and (b) into $(*)$ yields:

$$\tilde{d}_s(\iota_{\mathfrak{L}}(ax), \iota_{\mathfrak{L}}(by)) \leq 2^N \cdot 2^{-(N+n+1)} + 2^N \cdot 2^{-(N+n+1)} = 2^{-n}$$

Hence the map is locally uniformly continuous with module β. □

Reminder:

We previously proved the following theorem: For some family of uniform spaces $\{(X_i, D_i)\}_{i\in I}$, a complete unform space $(\tilde{Y}, \tilde{D}_Y)$ and a (locally) uniformly continuous map $f : \prod_{i\in I} X_i \to \tilde{Y}$, there exists a unique (locally) uniformly continuous extension $\tilde{f} : \prod_{i\in I} \tilde{X}_i \to \tilde{Y}$, i.e. the following diagram commutes:



Reminder:

The above map $\tilde{f}$ is an extension of f in the following sense. We can canonically extend f to the map:

$$\hat{f} : \prod_{i\in I} X_i \Bigg|_{\vec{\iota}\,(\prod X_i)} \to \tilde{Y}, \quad \hat{f}(\vec{\iota}\,(x)) := f(x)$$

Then $\tilde{f}$ is the unique (locally) uniformly continuous extension of $\hat{f}$ to $\prod_{i\in I} \tilde{X}_i$. Henceforth we will simply call $\tilde{f}$ the extension of f.

Lemma 4:

Given a family of uniform spaces $\{(X_i, D_i) \mid i \in I\}$, a complete uniform space $(\tilde{Y}, \tilde{D}_Y)$ and (locally) uniformly continuous maps $f, g : \prod_{i\in I} X_i \to \tilde{Y}$ and $\tilde{f}, \tilde{g} : \prod_{i\in I} \tilde{X}_i \to \tilde{Y}$, such that the following holds:

$$\tilde{f} \circ \vec{\iota} = f \quad \wedge \quad \tilde{g} \circ \vec{\iota} = g$$

Then the following holds:

$$\tilde{f} = \tilde{g} \quad \iff \quad f = g$$

Proof:

" $\Longrightarrow$ ": Let $\tilde{f} = \tilde{g}$. Then:

$$\tilde{f} \circ \vec{\iota} = \tilde{g} \circ \vec{\iota} \implies f = g$$

" $\Longleftarrow$ ": Let $f = g$. Then $\tilde{f}, \tilde{g}$ are both (locally) uniformly continuous extensions of $f = g$. By the uniqueness of the extension, we get:

$$\tilde{f} = \tilde{g}$$

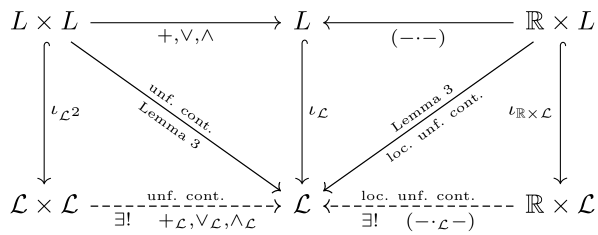
□

Prop 1:

The completion $(\mathcal{L}, \tilde{d}_s)$ of (L, d_s) is a vector lattice.

Proof:

By Lemma 3 the maps $\iota_{\mathcal{L}} \circ +$, $\iota_{\mathcal{L}} \circ \vee$ and $\iota_{\mathcal{L}} \circ \wedge$ are uniformly continuous and $\iota_{\mathcal{L}} \circ (- \cdot -)$ is locally uniformly continuous. By the above theorem, we can extend these maps. This yields the following commutative diagram:



We now have to show that these operations fulfill the axioms of a vector lattice. This will follow from lemma 4. We will show this for one but omit the rest:

$$\forall f, g, h \in \mathfrak{L} : (f +_{\mathfrak{L}} h) \vee_{\mathfrak{L}} (g +_{\mathfrak{L}} h) =_{\mathfrak{L}} f \vee_{\mathfrak{L}} g +_{\mathfrak{L}} h$$

To use Lemma 4 we define the following maps:

$$\varphi(x, y, z) := \iota_{\mathcal{L}}((x + z) \vee (y + z)) \qquad \tilde{\varphi}(f, g, h) := (f +_{\mathfrak{L}} h) \vee_{\mathfrak{L}} (g +_{\mathfrak{L}} h)$$
$$\psi(x, y, z) := \iota_{\mathcal{L}}(x \vee y + z) \qquad \tilde{\psi}(f, g, h) := f \vee_{\mathfrak{L}} g +_{\mathcal{L}} h$$

Here we have $\varphi, \psi : L^3 \to \mathcal{L}$ and $\tilde{\varphi}, \tilde{\psi} : \mathcal{L}^3 \to \mathcal{L}$. We now have:

$$\begin{aligned}
\tilde{\varphi}(\iota_{\mathfrak{L}}(x), \iota_{\mathfrak{L}}(y), \iota_{\mathfrak{L}}(z)) &= (\iota_{\mathfrak{L}}(x) +_{\mathfrak{L}} \iota_{\mathfrak{L}}(z)) \vee_{\mathfrak{L}} (\iota_{\mathfrak{L}}(y) +_{\mathfrak{L}} \iota_{\mathfrak{L}}(z)) \\
&= \iota_{\mathfrak{L}}(x + z) \vee_{\mathfrak{L}} \iota_{\mathfrak{L}}(y + z) = \iota_{\mathfrak{L}}((x + z) \vee (y + z)) \\
&= \varphi(x, y, z)
\end{aligned}$$

Thus we have $\tilde{\varphi} \circ \iota_{\mathcal{L}^3} = \varphi$. By the same argument we get $\tilde{\psi} \circ \iota_{\mathcal{L}^3} = \psi$. Since L is a vector lattice we have:

$$\varphi = \psi \xRightarrow{\text{Lemma 4}} \tilde{\varphi} = \tilde{\psi}$$

Repeating this argument for the other axioms yields the desired result. $\square$

Lemma 5:

The maps $E : L \to \mathbb{R}$ and $\| \cdot \| : L \to \mathbb{R}$ are uniformly continuous.

Proof:

Let $x, y \in L$ be arbitrary. Then we have:

(1) From $x \leq |x - y| + y$ we get:

$$E(x) \leq E(|x - y|) + E(y) = \|x - y\| + E(y) = d_s(x, y) + E(y)$$

Thus $E(x) - E(y) \leq d_s(x, y)$ and by symmetry we get:

$$|E(x) - E(y)| \leq d_s(x, y)$$

(2) From $\|x\| \leq \|x - y\| + \|y\|$ we get:

$$\|x\| \leq \|x - y\| + \|y\| = d_s(x, y) + \|y\|$$

Thus $\|x\| - \|y\| \leq d_s(x, y)$ and by symmetry we get:

$$|\|x\| - \|y\| \leq d_s(x, y)$$

□

Def 4:

Let $\int : \mathcal{L} \to \mathbb{R}$ be the unique extension of E to $\mathcal{L}$. We call $\int f$ the integral of $f \in \mathcal{L}$. Furthermore let $\| \cdot \|_{\mathcal{L}} : \mathcal{L} \to \mathbb{R}$ be the unique extension of $\| \cdot \|$ to $\mathcal{L}$. We call $\| \cdot \|_{\mathcal{L}}$ the norm on $\mathcal{L}$.

Lemma 6:

For all $f, g \in \mathcal{L}$ and $a \in \mathbb{R}$ the following holds:

(1) $\int (f +_{\mathcal{L}} g) = \int f + \int g$ and $\int (a \cdot_{\mathcal{L}} f) = a \int f$

(2) $0 \leq f \implies 0 \leq \int f$

(3) $\|f\|_{\mathcal{L}} = \int |f|$ and $\tilde{d}_s(f, g) = \|f - g\|_{\mathcal{L}}$

Proof:

(1) We have the following equalities:

$$\begin{aligned} \int (\iota_{\mathfrak{L}}(x) +_{\mathfrak{L}} \iota_{\mathfrak{L}}(y)) &= \int \iota_{\mathfrak{L}}(x+y) = E(x+y) \\ &= E(x) + E(y) = \int \iota_{\mathfrak{L}}(x) + \int \iota_{\mathfrak{L}}(y) \\ \int (a \cdot_{\mathfrak{L}} \iota_{\mathfrak{L}}(x)) &= \int \iota_{\mathfrak{L}}(ax) = E(ax) = aE(x) = a \int \iota_{\mathfrak{L}}(x) \end{aligned}$$

Lemma 4 then yields the desired result.

(2) For all $x \in L$ we have $0 \leq x^+$ and thus:

$$0 \leq E(x^+) = \int (\iota_{\mathfrak{L}}(x))^+ \Leftrightarrow \max\left\{0, \int (\iota_{\mathfrak{L}}(x))^+\right\} = \int (\iota_{\mathfrak{L}}(x))^+$$

By Lemma 4 we then get:

$$\forall f \in \mathfrak{L} : \max\left\{0, \int f^+\right\} = \int f^+ \Leftrightarrow 0 \leq \int f^+$$

The desired result then follows from:

$$0 \leq f \implies f = f^+ \implies 0 \leq \int f^+ = \int f$$

(3) We have the following equalities:

$$\begin{aligned} \|\iota_{\mathfrak{L}}(x)\|_{\mathfrak{L}} = \|x\| = E(|x|) = \int (\iota_{\mathfrak{L}}(|x|)) = \int |\iota_{\mathfrak{L}}(x)| \\ \tilde{d}_s(\iota_{\mathfrak{L}}(x), \iota_{\mathfrak{L}}(y)) = d_s(x, y) = \|x - y\| = E(|x - y|) = \int \iota_{\mathfrak{L}}(|x - y|) \\ = \int |\iota_{\mathfrak{L}}(x) - \iota_{\mathfrak{L}}(y)| = \|\iota_{\mathfrak{L}}(x) - \iota_{\mathfrak{L}}(y)\|_{\mathfrak{L}} \end{aligned}$$

Lemma 4 then yields the desired result. □