1 Approximation of square roots

... same as in the script ...

2 Cauchy sequences

As we saw in the last section, we can define a Cauchy sequence of rational numbers that does not converge to a rational number. Therefore, we shall view a real as a Cauchy sequence of rationals with a separately given modulus. This modulus witnesses that the points of the sequence become arbitrarily close, thus giving rise to a constructive notion of reals.

Definition 2.1. A real number x is a pair $((a_n)_{n\in\mathbb{N}}, M)$ with $a_n \in \mathbb{Q}$ and $M : \mathbb{Z}^+ \to \mathbb{N}$ such that $(a_n)_n$ is a Cauchy sequence with modulus M, that is

$$|a_n - a_m| \leq \frac{1}{2^p} \quad \text{for } n, m \geq M(p)$$

and M is weakly increasing, meaning $M(p) \leq M(q)$ for $p \leq q$. M is called the Cauchy modulus of x.

We shall loosely speak of a real $(a_n)_n$ if the Cauchy modulus M is clear from the context or inessential. Every rational a is tacitly understood as the real represented by the constant sequence $a_n = a$ with the constant modulus $M(p) = 0$.

It is certainly possible to find multiple Cauchy sequences and moduli describing the same real number. Hence, it is important to define an equivalence of reals capturing this intuitive notion. For technical reasons we will do so by first defining the order relations and then deriving equivalence from them. This way, many properties of this equivalence will be a consequence of the same properties of the $\leq$ relation, simplifying the proofs.

3 Nonnegative and positive reals

Firstly, we define what it means for a real x to be positive ($x > 0$) and to be nonnegative ($x \geq 0$). Note that being positive carries a computational value, as it should be possible to fit a small ball between zero and the number, which will be part of our definition.

Definition 3.1. A real $x := ((a_n)_n, M)$ is called *nonnegative* (written $x \in \mathbb{R}^{0+}$) if

$$-\frac{1}{2^p} \leq a_{M(p)} \quad \text{for all } p \in \mathbb{Z}^+.$$

It is *p-positive* (written $x \in_p \mathbb{R}^+$, or $x \in \mathbb{R}^+$ if p is not needed) if

$$\frac{1}{2^p} \leq a_{M(p+1)}.$$

The following description of a real being p-positive makes the geometric intuition of fitting a ball between zero and the number explicit.

Lemma 3.1 (RealPosChar)**.** *For a real $x := ((a_n)_n, M)$ with $x \in_p \mathbb{R}^+$ we have*

$$\frac{1}{2^{p+1}} \le a_n \quad \textit{for } n \ge M(p+1).$$

Conversely, from $\exists_{n_0} \forall_{n \ge n_0} (\frac{1}{2^q} \le a_n)$ we can infer $x \in_{q+1} \mathbb{R}^+$.

Proof. We shall first assume that $x \in_p \mathbb{R}^+$, that is $\frac{1}{2^p} \le a_{M(p+1)}$. Let $n_0 := M(p+1)$ and $n \ge n_0$. We have to show $\frac{1}{2^{p+1}} \le a_n$. Using the assumption of x being p-positive, we get

$$\begin{aligned} \frac{1}{2^{p+1}} &= \frac{1}{2^p} - \frac{1}{2^{p+1}} \\ &\le -\frac{1}{2^{p+1}} + a_{M(p+1)} \\ &= -\frac{1}{2^{p+1}} + (a_{M(p+1)} - a_n) + a_n \end{aligned}$$

Note that $n \ge M(p+1)$, thus by the defining property of $(x_n)_n$ being a Cauchy sequences we get

$$(a_{M(p+1)} - a_n) \le |a_{M(p+1)} - a_n| \le \frac{1}{p+1}$$

Putting it all together one has $\frac{1}{2^{p+1}} \le a_n$.

Conversely, assume that there is a natural number n_0 such that $\frac{1}{2^q} \le a_n$ for $n \ge n_0$. We aim to show $x \in_{q+1} \mathbb{R}^+$, that is $\frac{1}{2^{q+1}} \le a_{M(q+2)}$. Let $n \ge \max(M(q+2), n_0)$,

$$\begin{aligned} \frac{1}{2^{q+1}} &< -\frac{1}{2^{q+2}} + \frac{1}{2^q} \\ &\le -\frac{1}{2^{q+2}} + a_n \end{aligned}$$

As before, $a_n - a_{M(q+2)} \le |a_{M(q+2)} - a_n| \le \frac{1}{2^{q+2}}$. Multiplying by -1 yields $-\frac{1}{2^{q+2}} \le a_{M(q+2)} - a_n$.

$$-\frac{1}{2^{q+2}} + a_n \le (a_{M(q+2)} - a_n) + a_n = a_n$$

This shows $x \in_{q+1} \mathbb{R}$, finishing the proof. □

Lemma 3.2 (RealNNegChar)**.** *For a real $x := ((a_n)_n, M)$ the following are equivalent:*

(a) $x \in \mathbb{R}^{0+}$

(b) $\forall_p \exists_{n_0} \forall_{n \ge n_0} (-\frac{1}{2^p} \le a_n)$

Proof. To show that (a) implies (b), assume $x \in \mathbb{R}^{0+}$ and let $p \in \mathbb{Z}^+$. We will show that $n_0 := M(p+1)$ satisfies the condition. Let $n \ge n_0$. By assumption $-\frac{1}{2^{p+1}} \le a_{M(p+1)}$, hence

$$\begin{aligned} -\frac{1}{2^p} &= -\frac{1}{2^{p+1}} - \frac{1}{2^{p+1}} \\ &\le -\frac{1}{2^{p+1}} + a_{M(p+1)} \\ &= -\frac{1}{2^{p+1}} + (a_{M(p+1)} - a_n) + a_n \end{aligned}$$

Again, using that $(x_n)_n$ is a Cauchy sequence and $n \geq M(p+1)$, it holds that

$$-\frac{1}{2^{p+1}} + (a_{M(p+1)} - a_n) + a_n \leq -\frac{1}{2^{p+1}} + \frac{1}{2^{p+1}} + a_n = a_n$$

We now proceed to show that (b) implies (a). For this we use a characterization of the $\leq$ relation on rational number, which states, given $p, p' \in \mathbb{Q}$, $p \leq p'$ is equivalent to $p \leq p' + \frac{1}{2^q}$ for all $q \in \mathbb{Z}^+$ (RatLeAllPlusToLe). Now assume (b), let $p, q \in \mathbb{Z}^+$ and $n \geq \max(n_0, M(p))$ with n_0 provided by (b) for q. Then

$$\begin{aligned} -\frac{1}{2^p} - \frac{1}{2^q} &\leq -\frac{1}{2^p} + a_n \\ &= -\frac{1}{2^p} + (a_n - a_{M(p)}) + a_{M(p)} \\ &\leq -\frac{1}{2^p} + \frac{1}{2^p} + a_{M(p)} = a_{M(p)} \end{aligned}$$

Again, the first step follows from our assumption and the last step uses that $(a_n)_n$ is a Cauchy sequence and $n \geq M(p)$. Since q was arbitrary, this inequality holds for all $q \in \mathbb{Z}^+$. Thus, $-\frac{1}{2^p} \leq a_{M(p)}$ for all $p \in \mathbb{Z}^+$ by RatLeAllPlusToLe, which is precisely the definition of $x \in \mathbb{R}^{0+}$. □

4 Addition, Negation and Absolute Value

We want to define $x < y$ for reals x, y if and only if $y - x \in \mathbb{R}^+$ and analogously for $\leq$. In order to do so, we will need to define the addition of two reals and the negation a real.

Definition 4.1. Given real number $x := ((a_n)_n, M)$ and $y := ((b_n)_n, N)$, we define $x + y$, $-y$ and $|x|$ as represented by the respective sequence $(c_n)_n$ of rational number with modulus K:

	c_n	$K(p)$
$x + y$	$a_n + b_n$	$\max(M(p+1), N(p+1))$
$-x$	$-a_n$	$M(p)$
$\lvert x\rvert$	$\lvert a_n\rvert$	$M(p)$

Lemma 4.1. *For reals x, y also $x + y$ (RealPlusReal), $-x$ (RealUMinusReal) and $|x|$ (RealAbsReal) are reals.*

Proof. The fact $-x$ is a real clearly follow from x being a real. In the case of $|x|$ use the inequality $||a| - |b|| \leq |a - b|$ for all $a, b \in \mathbb{Q}$. For $x + y$ let $n, m \geq K(p) = \max(M(p+1), N(p+1))$.

$$\begin{aligned} |c_n - c_m| &= |x_n + y_n - (x_m + y_m)| \\ &\leq |x_n - x_m| + |y_n - y_m| \\ &\leq \frac{1}{p+1} + \frac{1}{p+1} = \frac{1}{p} \end{aligned}$$

□

Note that in the definition of the modulus of $x + y$ one has to increase p by one. This stems from the observation, that, in general, the "errors" of the sequences add up, as seen in the proof.

5 Comparison of reals

Definition 5.1. Let x, y be reals, we write $x \le y$ for $y - x \in \mathbb{R}^{0+}$ and $x < y$ for $y - x \in \mathbb{R}^{+}$.

Unwinding the definitions yields that $x \le y$ is to say that for every $p \in \mathbb{Z}^+$ we have $a_{K(p)} \le b_{K(p)} + \frac{1}{2^p}$ with $K(p) := \max(M(p+1), N(p+1))$. Furthermore, $x < y$ is a shorthand for the presence of p with $a_{K(p+1)} + \frac{1}{2^p} \le b_{K(p+1)}$ with K as before; we then write $x <_p y$ whenever we want to call these witnesses.

Lemma 5.1 (RealApprox)**.** $\forall_{x,p} \exists_a (|x - a| \le \frac{1}{2^p})$

Proof. ... same as in the script ... □

Lemma 5.2 (RealLeChar)**.** *For reals* $x := ((a_n)_n, M)$*,* $y := ((b_n)_n, N)$ *the following are equivalent:*

(a) $x \le y$

(b) $\forall_p \exists_{n_0} \forall_{n \ge n_0} (a_n \le b_n + \frac{1}{2^p})$

Proof. This is an immediate consequence from RealNNegChar. □

Lemma 5.3 (RealLtChar)**.** *For reals* $x := ((a_n)_n, M)$*,* $y := ((b_n)_n, N)$ *with* $x <_p y$ *we have*

$$a_n + \frac{1}{2^{p+1}} \le b_n \quad \text{for } n \ge \max(M(p+2), N(p+2))$$

Conversely, from $\exists_{n_0} \forall_{n \ge n_0} (a_n + \frac{1}{2^q} \le b_n)$ *we can infer* $x <_{q+1} y$*.*

Proof. As before, this follows from RealPosChar. □

Lemma 5.4. *For reals* x, y, z*,*

$$\begin{array}{cc} x \le x & x \not< x \\ x \le y \to y \le z \to x \le z & x < y \to y < z \to x < z \\ x \le y \to x + z \le y + z & x < y \to x + z < y + z \end{array}$$

Proof. These properties can easily be seen using the characterizations from above. For example, to prove transitivity of $<$, note that for reals x, y, z with $x <_p y$ and $y <_q z$, there is $n_0 \in \mathbb{N}$ such that for all $n \ge n_0$ one has $a_n + \frac{1}{2^{p+1}} + \frac{1}{2^{q+1}} \le b_n + \frac{1}{2^{q+1}} \le c_n$. □

Here we have left out ... (same as in the script till Chapter 7.)

6 Equality of reals

Definition 6.1. Two reals x, y are called equivalent (or equal and written $x = y$, if the context makes clear what is meant), if $x \leq y$ and $y \leq x$.

Remark. Using this definition of equality, it immediately follows that being nonngeative is compatible with equality. Compatibility with < and being positive follows from ... script Lemma 6.4 ...

Lemma 6.1 (RealEqChar)**.** *For reals $x := ((a_n)_n, M)$, $y := ((b_n)_n, N)$ the following are equivalent:*

(a) $x = y$

(b) $\forall_p \exists_{n_0} \forall_{n \geq n_0} (|a_n - b_n| \leq \frac{1}{2^p})$

Proof. First assume (a). This by definition means $x \leq y$ and $y \leq x$. Let $p \in \mathbb{Z}^+$ using RealLeChar we get $n_0, n_0' \in \mathbb{N}$ such that for all $n \geq \max(n_0, n_0')$ one has $a_n \leq b_n + \frac{1}{2^p}$ and $b_n \leq a_n + \frac{1}{2^p}$. This shows $|a_n - b_n| \leq \frac{1}{2^p}$. In fact, we only used equivalences, concluding the proof. □

Remark (RealSeqEqToEq). ... same as in the script ...

Lemma 6.2 (RealEqTrans)**.** *Equality between reals is transitive.*

Proof. Follows from the transitivity of the $\leq$ relation. □

7 The Archimedian property

For every function on the reals we certainly want compatibility with equality. This however is not always the case; here is an important example.

Lemma 7.1 (RealBound)**.** *For every real $x := ((a_n)_n, M)$ we can find p_x such that $|a_n| \leq 2^{p_x}$ for all n.*

Proof. Let $n_0 := M(1)$ and p_x be such that $\max\{\, |a_n| \mid n \leq n_0 \,\} + \frac{1}{2} \leq 2^{p_x}$. If $n \leq n_0$, then by choice of p_x it holds that $|a_n| \leq 2^{p_x}$. Now if $n > n_0 = M(1)$ then by $(a_n)_n$ being a Cauchy sequence we have

$$|a_n| = |(a_n - a_{n_0}) + a_{n_0}| \leq |a_n - a_{n_0}| + |a_{n_0}| \leq \frac{1}{2} + |a_{n_0}| \leq 2^{p_x}$$

□

Clearly this assignment of p_x to x is not compatible with equality.

8 Multiplication and inverse

Using the Archimedean property we are able to define multiplication of reals.

Definition 8.1. Given real number $x := ((a_n)_n, M)$ and $y := ((b_n)_n, N)$, we define $x \cdot y$ and provided $|x| > 0$ also $\frac{1}{x}$ as represented by the respective sequence $(c_n)_n$ of rational number with modulus K:

	c_n	$K(p)$
$x \cdot y$	$a_n \cdot b_n$	$\max(M(p+1+p_y), N(p+1+p_x))$
$\frac{1}{x}$ for $\lvert x\rvert \in_q \mathbb{R}^+$	$\begin{cases} \frac{1}{a_n} & \text{if } a_n \neq 0 \\ 0 & \text{if } a_n = 0 \end{cases}$	$M(2(q+1)+p)$

where p_x and p_y are provided by RealBound.

Lemma 8.1. *For reals x, y also $x \cdot y$ and provided that $0 <_q |x|$ also $\frac{1}{x}$ are reals.*

Proof. ... same as in the script ... □

9 Compatibility

Lemma 9.1. *For reals x, y, z,*

$$x \le y \to 0 \le z \to x \cdot z \le y \cdot z \quad x < y \to 0 < z \to x \cdot z < y \cdot z$$

Proof. Follows from RealLtChar and RealLeChar. □

Lemma 9.2. *For reals x, y, z*

$$x + (y + z) = (x + y) + z \qquad x \cdot (y \cdot z) = (x \cdot y) \cdot z$$
$$x + 0 = 0 \qquad x \cdot 1 = x$$
$$x + (-x) = 0 \qquad 0 < |x| \to x \cdot \frac{1}{x} = 1$$
$$x + y = x + y \qquad x \cdot y = y \cdot x$$

$$x \cdot (y + z) = x \cdot y + x \cdot z$$

Proof. ... same as in the script ... □

Lemma 9.3. *The functions $x + y, -x, |x|, x \cdot y$ and (provided that $|x| \in_q \mathbb{R}^+$) also $\frac{1}{x}$ are compatible with equality.*

Proof. The compatibility with addition follows from Lemma **??**. For the compatibility with multiplication we unfortunately can't use Lemma **??**, since constructively the case distinction $0 \le z$ or $z \le 0$ is not possible as seen before. Nevertheless, using RealEqChar this can quickly be shown: Assume $x = y$, we want to show $x \cdot z = y \cdot z$. Pick an arbitrary $p \in \mathbb{Z}^+$ and let p_z be provided by RealBound for z. Using RealEqChar pick n_0 such that $n \ge n_0$ implies $|a_n - b_n| \le \frac{1}{2^{p+p_z}}$. Now,

$$|a_n c_n - b_n c_n| \le |a_n - b_n| \cdot 2^{p_z} \le \frac{1}{2^{p+p_z}} \cdot 2^{p_z} \le \frac{1}{2^p}$$

This shows $x \cdot z = y \cdot z$.

For compatibility with absolute value let $x = y$ and note that

$$a_n \leq b_n + \frac{1}{2^p} \Rightarrow |a_n| \leq |b_n + \frac{1}{2^p}| \leq |b_n| + \frac{1}{2^p}$$

Thus, the statement follows from compatibility with $\leq$ and RealLeChar using the same n_0. For unary division also use RealEqChar. □

Lemma 9.4. *For a real x from $0 \leq x$ and $0 \leq -x$ we can infer $x = 0$.*

Proof. ... same as in script (Lemma 5.5 (c)) ... □

Lemma 9.5. *For reals x, y from $x \cdot y = 1$ we can infer $0 < |x|$.*

Proof. Using RealBound Pick p such that $|b_n| \leq 2^p$ for all n. By RealEqChar we can choose n_o such that for all $n \geq n_0$ we have $1 - a_n b_n \leq \frac{1}{2}$, hence $\frac{1}{2} \leq a_n b_n$. Then we have $\frac{1}{2} \leq a_n 2^p$, and thus $\frac{1}{2^{p+1}} \leq |a_n|$. □