Konstruktive Analysis WS 23/24 Skript: new.scm

Important properties of the exponential function

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We want to verify the proposition

Proposition. *(a)* $\forall_{x \in \mathbb{R}} 0 \leq \exp(x)$

(b) $\forall_{x \in \mathbb{R}} 0 < \exp(x)$

(c) $\forall_{x \in \mathbb{R}} \exp(-x) = \exp(x)^{-1}$

(d) $\forall_{k \in \mathbb{Z}} \exp(k) = e^k$

with Minlog. But first we will proof the proposition. The following proof is almost the same as the one in Constructive analysis with witnesses by Helmut Schwichtenberg (Version 2023/24).

Proof. First we note that due to the functional equation we have

$$\exp(x)\exp(-x) = \exp(x - x) = \exp(0) = 1$$

(a) Since our goal is stable, we can distinguish cases (i) $0 \leq x$ and (ii) $\neg(0 \leq x)$. In case (i) we clearly have $\exp(x) \geq 1$. In case (ii) we have $x \leq 0$. We then have $\exp(-x) \geq 1 > 0$. Therefore it follows:

$$0 \leq \frac{1}{\exp(-x)} = \exp(x)$$

The equality on the right is implied by the functional equation at $x, -x$.

(b) Pick $p \in \mathbb{Z}^+$ with $\exp(-x) \leq p$. Then $1 = \exp(x)\exp(-x) \leq \exp(x)p$ by (a). Hence the claim.

(c) We can obviously derive the goal from the functional equation together with (b)

(d) We use induction on n. Clearly $\exp(0) = 1 = e^0$; for $n \mapsto n + 1$:

$$\exp(n + 1) = \exp(n)\exp(1) = e^n \cdot e = e^{n+1}$$

and for $k < 0$:

$$\exp(k) = \frac{1}{\text{exp(-k)}} = \frac{1}{e^{-k}} = e^k$$

□

We will now give some comments to the proofs of our proposition found in `new.scm`. We start with the important lemmas `RealECompat` and `RealEFunEqZero` and give then some context for the proofs of the parts of our proposition: (a) `RealELeZero`, (b) `RealEPos`, (c) `RealEInvEqMinus`, (d)(i) `RealEToExpNat` and (d)(ii) `RealEToExpNeg`.

1 RealECompat

`RealECompat` is the statement

$$\forall_{x,y}(x =_{\mathbb{R}} y \to \exp(x) =_{\mathbb{R}} \exp(y))$$

With $=_{\mathbb{R}}$ we mean equality of real numbers modulo representation. Intuitively this statement says that the result of exp is independent of the representation of its input. This statement appears to be trivial since in usual mathematics we build all our methods modulo representation. Proofing it rigorously is non the less non-trivial.

We verify this statement by using `RealConvLimUniq`:

$$\forall_{xs,y,M,z,N}(xs \to_M y) \to (xs \to_N z) \to y =_{\mathbb{R}} z$$

$xs \to_M y$ stands for xs converging to y with modul M. The corresponding predicate is called `RealConvLim` in Minlog. This statement essentially says that if a sequence converges, its limit is unique. To proof our goal, we instantiate $y := \exp(x)$, $z := \exp(y)$ and for xs we choose the exponential series at x. For M we choose the modul of the exponential series at x, `cREMod x` and for N we take the modulo of the exponential series at y, `cREMod y`. The condition $xs \to_M y = \exp(x)$ of `RealConvLimUniq` only needs the convergence of the exponential series, which is given by `RealEConvLim`. We proof $xs \to_N \exp(y)$ by using `RealConvLimCompat`:

$$\forall_{xs,x,M,ys,y,N}(\forall_n xs_n =_{\mathbb{R}} ys_n \to x =_{\mathbb{R}} y \to \forall_p M(p) =_{\mathbb{N}} N(p) \to (xs \to_M x) \to (ys \to_N y))$$

This theorem states that the representation of sequence, modul and limit are not relevant for convergence. It fits therefore perfectly our need, since we have $ys \to_N \exp(y)$ by the convergence of the exponential series `RealEConvLim` and we can easily proof $\forall_n xs_n =_{\mathbb{R}} ys_n$ due to $x =_{\mathbb{R}} y$ using compatibility of equality with simple operations on $\mathbb{R}$.

2 RealEFunEqZero

The statement `RealEFunEqZero`

$$\forall_x(x \in \mathbb{R} \to 1 =_{\mathbb{R}} \exp(x) \cdot \exp(-x))$$

is a simple specialization of the functional equation `RealEFunEq`

$$\forall_{x,y}(x \in \mathbb{R} \to y \in \mathbb{R} \to \exp(x+y) =_{\mathbb{R}} \exp(x) \cdot \exp(y))$$

The rigorous proof of `RealEFunEqZero` is yet again non-trivial, since we need to proof (i) $1 =_{\mathbb{R}} \exp(0)$ and (ii) $\exp(x + (-x)) = \exp(0)$. For (ii) we can use `RealECompat`. For (i) we yet again use `RealConvLimUniq` with $y := 1$, $z := \exp(0)$ and for xs we choose the exponential series at 0. For M we choose the constant modul 0. For N we choose the modul of the exponential series at 0. We then proof $xs \to_M 1$ by stepping through the definition of the predicate. We get $xs \to_N \exp(0)$ by the convergence of the exponential series, `RealEConvLim`.

3 RealELeZero

This statement is part (a) of our proposition and states that the exponential function is everywhere non-negative. This statement will be an integral part of the proof of (b).

From the proof of (a) one easily notices that we use the fact $\forall_x 0 \leq x \to 1 \leq \exp(x)$ twice: First we use it in the first case of the case distinction and we also use it in the second case when we deduce $1 \leq \exp(-x)$ from $x \leq 0$. It is therefore good practice to define this statement as a lemma.

We proof the lemma $\forall_x 0 \leq x \to 1 \leq \exp(x)$ by proofing that the elements of the constant sequence of 1 are everywhere smaller than the ones of the exponential series at x. We use the theorem `RealConvLimLe`

$$\forall_{x,y,xs,ys,M,N}(\forall_n xs_n \leq ys_n \to xs \to_M x- > ys \to_N y- > x \leq y)$$

for this.

Further we want to proof by cases in (a) on the order of 0 and x. Since $\leq$ is not decidable on $\mathbb{R}$, we need to argue by contradiction. To use proof by contradiction in a constructive setting we utilize the stability of our goal $0 \leq \exp(x)$. We use`RealLeStab`

$$\forall_{x,y}(((x \leq y \to \mathbf{F}) \to \mathbf{F}) \to x \leq y)$$

and `RealLeCases`

$$\forall_{x,y}(x \in \mathbb{R} \to y \in \mathbb{R} \to (x \leq y \to \mathbf{F}) \to (y \leq x \to \mathbf{F}) \to \mathbf{F})$$

In the first case $0 \leq x$ we can simply use the lemma. In the second case $x \leq 0$ we argue basically the same way like we do in the proof from before, but we have the following larger complications: We need to use `RealPosToZeroLeUDiv`

$$\forall_{x,p}(x \in \mathbb{R} \to 0 <_p x \to 0 \leq 1/_p x)$$

to show $0 \leq \frac{1}{\exp(-x)}$ and `RealTimesUDivR`

$$\forall_{x,p}(x \in \mathbb{R} \to 0 <_p x \to x * 1/_p x =_{\mathbb{R}} 1)$$

to show $\frac{1}{\exp(-x)} = \exp(x)$ from the functional equation. Here $0 <_p x$ stands for x being positive with modul p and $1/_p x$ is the inverse of x where x is positive with modul p. This modul for the positivity of x is needed for the modul of the real number $1/_p x$.

4 RealEPos

This part of the proposition states that the exponential series is positive everywhere. As a formula we get:

$$\forall_x (x \in \mathbb{R} \to \exists_p 0 <_p \exp(x)$$

Since we have an existence quantifier in the formula the theorem has computational content. For every x we have a lower bound on the value of $\exp(x)$ where the lower bound is always greater than zero.

In our proof we set $p := n + 2$, where n is a number so that $\exp(-x) \leq 2^n$. Those values are determined by `RealLeToPos` and `RealLeBound`. Apart from determining the correct p the proof is pretty straightforward.

5 RealEInvEqMinus

As mentioned before we need a modul to define the inverse of a real number. Therefore we yet again need to find a modul p. Part (c) of the proposition has the formula

$$\forall_x (x \in \mathbb{R} \to \exists_p \exp(-x) =_{\mathbb{R}} 1/_p \exp(-x))$$

We get the modul p directly from part (b) and can proof the rest of the statement pretty easily by using `RealTimesUDivR` and `RealEFunEqZero`

6 RealEToExpNat

Since exponentiation of real numbers by integer is not defined we split part (d) of the proposition in two parts. In this part we will show the statement for alle natural numbers. In the second part we will show the statement for all non-positive integers.

In this part we use simple induction on the natural numbers together with the (specialized) functional equation just like in the proof of (d) to verify the statement.

7 RealEToExpNeg

In this second part of (d) of the proposition we show the statement of (d) for all non-positive integers:

$$\forall_n (\exists_p \exp(-n) =_{\mathbb{R}} (1/_p \exp(1)) **n)$$

Since we do not have exponentiation of real numbers by negative integers we use two steps. We simply raise $\frac{1}{\exp(0)}$ to the power of $-k \in \mathbb{N}$ instead of raising $\exp(0)$ to the power of k. But we get again the problem that we need a witness for the inverse of $\exp(0)$, but we can simply use the one from part (c), `RealEInvEqMinus`.

We therefore only have to proof $\exp(k) =_{\mathbb{R}} (\frac{1}{e})^{-k}$ for all $-k \in \mathbb{N}$. Despite that we still can't write the proof as quick as we did earlier. We could use (c), but that would only help us if we wanted to proof $\exp(k) =_{\mathbb{R}} \frac{1}{e^{-k}}$ for $-k \in \mathbb{N}$. We therefore need to use induction again to proof $\exp(k) =_{\mathbb{R}} (\exp(-1))^{-k}$ for $-k \in \mathbb{N}$.