1 Introduction

This document explains, in which way the Completeness of the real numbers (i.e. that every Cauchy-sequence of reals converges to a real number) can be proofed, using the proof-assistant MINLOG. The bits of code appearing in this text are extracted from the file `librseq.scm` starting from the header `1.Completion of Reals`. The noteworthly definitons and theorems are:

```
RatCauchyConvMod (in file "rat.scm")
RCauchy, RealCauchy
RealLim, cRLim
RealLimReal
RealCompleteAux1, RealCompleteAux1
RealCauchyConvMod
RealComplete
```

where `RealComplete` is the final theorem, proving the Completeness of the reals.
Throughout the text, we assume the reader to have suffient knowledge about the implementation of $\mathbb{N}, \mathbb{P}, \mathbb{Q}$ and $\mathbb{R}$ in MINLOG, as well as basic arithmetical statements for these. Information about these can be found in the corresponding texts or library-files of MINLOG.

2 RatCauchyConvMod

Recall, that in `rea.scm` we have introduced the natural notion of reals by an inductive predicate `Real` of arity $\mathbb{R} = (\mathbb{N} \to \mathbb{Q}) \times (\mathbb{P} \to \mathbb{N})$. In fact, $x \in$`Real` iff the sequence of rationals $((a_n)_n, M)$, defining x, is Cauchy-sequence and M a monotone function.
Tackling the Completeness of $\mathbb{R}$, one usually starts by proveing the "well-definedness" of the reals by Cauchy-sequences and moduli, in the sense, that the Cauchy-sequence of rationals $((a_n)_n, M$ defining $x \in$`Real` converges to x with modulus M. We give a natural proof of this.

Theorem 2.1 (RatCauchyConvMod)
For any $x = ((a_n)_n, M) \in \mathbb{R}$*:* $x \in$`Real` $\Rightarrow \forall p, n \geq M(p) : \ |a_n - x| \leq \frac{1}{2^p}$

Proof:
Let $p \in \mathbb{P}$ and $n \geq M(p)$.
The definitions in `rea.scm` state, that $|a_n - x| = ((|a_n - a_m|)_{m \in \mathbb{N}}, N)$,
with $N(p) :=\max(M(p+1), 0(p+1))$, where 0 is the 0-modulus from the inclusion of $\mathbb{Q}$ in $\mathbb{R}$. Moreover, $|a_n - x| \leq \frac{1}{2^p}$ in $\mathbb{R}$ iff $|a_n - a_{K(q)}| \leq \frac{1}{2^p} + \frac{1}{2^q}$ in $\mathbb{Q}$ for every q. Here, $K :=\max(N(q), 0(q))$.
It therefore suffices to prove the later statement about rationals.
Let $q \in \mathbb{P}$. We can now use that $((a_n)_n, M)$ is a Cauchy-sequence with monotone M, since $x \in$`Real`. Hence, we know that $|a_n - a_m| \leq \frac{1}{2^r}$, whenever $n, m \geq M(r)$.
Suppose $q \geq p$, then we instantly obtain $|a_n - a_{K(q)}| \leq \frac{1}{2^p} \leq \frac{1}{2^p} + \frac{1}{2^q}$, as $K(q) \geq M(p)$.
Conversely, suppose $q < p$, then we'll see, that $|a_n - a_{K(q)}| \leq \frac{1}{2^q} \leq \frac{1}{2^p} + \frac{1}{2^q}$, because $q + 1 \leq p$ gives $n \geq M(p) \geq M(q+1) = K(q)$. □

As usual, the MINLOG-implementation follows this natural proof. The statement of `RatCauchyConvMod` then is the following

```
(set-goal "all as,M,p,n(Real(RealConstr as M) -> M p<=n
-> abs(as n+ ~(RealConstr as M))<<=(1#2**p))")
```

We can then fix variables and make the assumptions avaible for the proof using the comand `(assume "as" "M" "p" "n" "Rx" "nBd")`. Hence, MINLOG asks us to prove

```
2:abs(as n+ ~(RealConstr as M))<<=(1#2**p)
```

from the assumptions

```
as   M   p   n   Rx:Real(RealConstr as M)  nBd:M p<=n
```

Now we recall that the theorem `RealLeChar2RealConstrFree` gives us the Definition of $\leq$ in $\mathbb{R}$, stateing: $\forall x = ((c_n)_n, C), y = ((d_n)_n, D) \in$`Real`, $\forall p\ \exists n\ \forall n0 \geq n :\ c_n 0 \leq d_n 0 + \frac{1}{2^p} \rightarrow x \leq y$. If we use `RealLeChar2RealConstrFree`, we then obtain the goals:

```
?^3:Real(abs(as n+ ~(RealConstr as M)))
?^4:Real(1#2**p)
?^5:all p0 exnc n0 all n1
(n0<=n1 -> (abs(as n+ ~(RealConstr as M)))seq n1<=(1#2**p)seq n1+(1#2**p0)
```

We will discuss `^?3` and `^?4` here, as they are simple applications of basic theorems in `rea.scm`. Using `(ng)` on `^?5`, then reduces to the goal:

```
 ?^10:all p0 exnc n0 all n1
(n0<=n1 -> abs(as n+ ~(as n1))<=(2**p0+2**p#2**p*2**p0)
```

Notice, that this can be proved using reasoning in $\mathbb{Q}$, only. Fixing the variable `p0`, one then easily sees, that $\frac{2^{p0}+2^p}{2^p 2^{p0}} = \frac{1}{2^p} + \frac{1}{2^{p0}}$. Hence, we can simplify our goal using with the comand

```
(simp (pf "(2**p0+2**p#2**p*2**p0)=((1#2**p)+(1#2**p0))"))
```

Now, MINLOG replaces the terms, which are equal according to the proposed equality, and asks aus to prove the newly constructed goal and the proposed equality

```
?^12:exnc n0 all n1(n0<=n1 -> abs(as n+ ~(as n1))<=(1#2**p)+(1#2**p0)
?^13:(2**p0+2**p#2**p*2**p0)=(1#2**p)+(1#2**p0)
```

`^?13` however, can be proved using simple arithmetics of $\mathbb{Q}$. Hence, we will omit a closer discussion of a proof of `^?13`.
For `^?12` one now proposes $M(p)$ as a possible option to take the place of $n0$ in the existence statement. This si done using the comand `(intro 0 (pt "M p"))`. MINLOG then renames $n1$ to $n0$. Fixin this variable and assuming `n0bd: M p`leq`n0`, we then obtain the goal:

```
?^15:abs(as n+ ~(as n0))<=(1#2**p)+(1#2**p0)
```

Now, this can obviously be proved using the Cauchy-property of `as`! Hence, with `(use "RatLeTrans" (pt "(1#2**p)+0"))` we kill the `(1#2**p0)` from our goal. Of course, this also gives us a new goal, namely to show that `(1#2**p)+0<=(1#2**p)+(1#2**p0)`. How to come up with a proof of this should be clear, however. With the Elimination-Axiom of `Cauchy`, we obtain the goals

```
?^18:Cauchy as M   ?^19:M p<=n   ?^20:M p<=n0
```

We instantly see that these goals can be proved using the assumptions `Rx`, `nBd` and `n0Bd`. □

3 Cauchy-Sequences in $\mathbb{R}$

Next, we'll need a notion of Cauchy-sequences with moduli for $\mathbb{R}$. Following the basic intuition from the natural proof, we can give a first definition by an inductive predicate.

Definition 3.1
`RCauchy`$:=\mu_I\ (\forall (x_n)_n, M\ (\forall p, n, m\ (n, m \geq M(p) \rightarrow |x_n - x_m| \leq \frac{1}{2^p}) \rightarrow I((x_n)_n, M))$

This gives us the following Introduction- and Elimination-Axioms for `RCauchy`:
$($`RCauchy`$)^+$: $\forall (x_n)_n, M\ (\forall p, n, m\ (n, m \geq M(p) \rightarrow |x_n - x_m| \leq \frac{1}{2^p}) \rightarrow$`RCauchy`$((x_n)_n, M))$
$($`RCauchy`$)^-$: $\forall (x_n)_n, M\ (\forall p, n, m\ (n, m \geq M(p) \rightarrow |x_n - x_m| \leq \frac{1}{2^p}) \rightarrow ((x_n)_n, M) \in X \rightarrow$`RCauchy`$\subset X$

As RCauchy is defined inductively, MINLOG automatically saves the axiom $(\texttt{RCauchy})^+$ as theorem RCauchyIntro. $(\texttt{RCauchy})^-$, in contrary, still needs to be proved. However, this is a standard procedure and we will not discuss it here.
Obviously,RCauchy doesn't keep track, whether the members of Cauchy-Sequence are constructive reals (not partial ones). Hence, it will be useful to have an extra notion which makes sure, that we're talking about concrete reals.

Definition 3.2
RealCauchy:=$\mu_I\ (\forall (x_n)_n, M\ (\forall n (x_n \in \texttt{Real}) \to M \in \texttt{Mon} \to \texttt{RCauchy}((x_n)_n, M)\ \to I((x_n)_n, M))$

From the Elimination-Axiom $(\texttt{RealCauchy})^-$, we can instantly derive the following partial eliminations:
RealCauchyToReals: $\forall (x_n)_n, M\ (((x_n)_n, M) \in \texttt{RealCauchy} \to \forall n\, (x_n \in \texttt{Real}))$
RealCauchyToMon: $\forall (x_n)_n, M\ (((x_n)_n, M) \in \texttt{RealCauchy} \to M \in \texttt{Mon})$
RealCauchyToRCauchy:$\forall (x_n)_n, M\ (((x_n)_n, M) \in \texttt{RealCauchy} \to (((x_n)_n, M) \in \texttt{RCauchy})$

4 A Natural Proof of RealComplete

Let's recall, how the Completeness of Reals is proved naturally, to get the idea of diagonalizing and to fix first notation.

Theorem 4.1 (RealComplete)
For any $((x_n)_n, M) \in$ RealCauchy, we can find a $x \in$Real, such that $((x_n)_n$ converges to x with modulus M uniquely, i.e. $\forall p,\ n \geq M(p)\, (|x_n - x| \leq \frac{1}{2^p})$.

Proof:
1.Step: Find a Candidate for x (cRLim)
By RealCauchyToReals we know that $x_n \in$Real for every n. Hence, we can pick a Cauchy-sequence $((a_k^{(n)})_k, N_n)$ of $\mathbb{Q}$, such that $x_n = ((a_k^{(n)})_k, N_n))$. This leaves us with a double indexed sequence $((a_k^{(n)})_k)_n$ and a sequence of moduli N_n. By the nature of Cauchy-sequences, we suspect that $(x_n)_n$ narrows down to a point x for $n \to \infty$. Therefore, we'd like to have a sequence of $\mathbb{Q}$, which follows $(x_n)_n$ down the line. Note, that RatCauchyConvMod states, that for all $k \geq N_n(p)$, we have $|a_k^{(n)} - x_n| \leq \frac{1}{2^p}$ for any p. Hence, setting $b_n := a_{N_n}^{(n)}(n)$, provides us with a sequence $((b_n)_n)$, that approaches $(x_n)_n$ with error at most $\frac{1}{2^n}$.
In consequence, if x were the limit of x_n, for any $n \geq$max$(M(p+1), p+1)$ we'd get

$$|b_n - x| \leq |b_n - x_n| + |x_n - x| = |a_{N_n(n)}^{(n)} - x_n| + |x_n - x| \leq \frac{1}{2^n} + \frac{1}{2^{p+1}} \leq \frac{1}{2^{p+1}} + \frac{1}{2^{p+1}} = \frac{1}{2^p}$$

Choosing $K(p) :=$max$(M(p+1), p+2)$ therefore gives the desired diagonal sequence $((b_n)_n, K)$. We can now choose our candidate for the limit as $x := ((b_n)_n, K)$. However, we'd still have to show that this is well-defined, i.e. that this x is a Real again.

2.Step: Show x$\in$ Real (RealLimReal)
By the Introduction-Axiom of Real, it therefore suffices to show, that $(b_n)_n$ is a Cauchy-Sequence with modulus K (RealCompleteAux2) and K is a monotone function.
The second part is obvious, since $K(p) =$max$(M(p+1), p+2)$ and $M \in$Mon.
The first part, however takes a bit more effort.
Here, we first prove the claim RealCompleteAux1: $\forall n\, (|b_n - x_n| \leq \frac{1}{2^n}$.
By Definition of $\leq_{\mathbb{R}}$, b_n and x_n we know that:

$$|b_n - x_n| = |a_{N_n(n)}^{(n)} - x_n| \leq \frac{1}{2^n} \Leftrightarrow |a_{N_n(n)}^{(n)} - a_{N_n(q)}^{(n)}| \leq \frac{1}{2^n} + \frac{1}{2^q}\ \forall q$$

But now $|a^{(n)}_{N_n(n)} - a^{(n)}_{N_n(q)}| \leq \begin{cases} \frac{1}{2^n} & \text{for } n \leq q \\ \frac{1}{2^q} & \text{for } n > q \end{cases} \leq \frac{1}{2^n} + \frac{1}{2^q} \forall q$. This gives the claim.
Now, let $p \in \mathbb{P}$ and $n, m \geq K(p)$. Then, since x_n is itself a Cauchy-sequence, we get

$$|b_n - b_m| \leq |b_n - x_n| + |x_n - x_m| + |x_m - b_m| \leq \frac{1}{2^n} + \frac{1}{2^{p+1}} + \frac{1}{2^m} \leq \frac{1}{2^{p+2}} + \frac{1}{2^{p+1}} + \frac{1}{2^{p+2}} = \frac{1}{2^p}$$

3.Step: $(x_n)_n$ converges to x with modulus M: (`RealCauchyConvMod`)
Let p be fixed and q any positive number. Also, let $n \geq M(p)$, $m \geq max(K(q+1), q+1, M(p))$. Obviously

$$|x_n - x| \leq |x_n - x_m| + |x_m - x| \leq |x_n - x_m| + |x_m - b_m| + |b_m - x|$$

Then, $|x_n - x_m| \leq \frac{1}{2^p}$, since $(x_n)_n$ is a Cauchy-sequence.
Moreover, $|x_m - b_m| \leq \frac{1}{2^{q+1}}$ by `RealCompleteAux1`.
And lastly, $|b_m - x| \leq \frac{1}{2^{q+1}}$. This is, because $((b_n)_n, K)$ is a Cauchy-sequence in $\mathbb{Q}$ by 2.Step(`RealCompleteAux2`), which defines x, and by `RatCauchyConvMod`, Cauchy-Sequences converge to the Real, which they define, within their Modulus .
Hence, we obtain, that

$$|x_n - x| \leq \frac{1}{2^p} + \frac{1}{2^{q+1}} + \frac{1}{2^{q+1}} = \frac{1}{2^p} + \frac{1}{2^q} \text{ for any q}$$

This means $|x_n - x| \leq \frac{1}{2^p}$, of course. □We will now examine, in which way this natural proof can be implemented in MINLOG

5 Step 1: Candidates for the Limit: RealLim and cRLim

As we saw, the first step in the proof of Completeness of Reals was to derive a candidate for the limit, to which the given Cauchy-Sequence $((x_n)_n, M)$ converges. This involved defining a diagonal sequence $(b_n)_n, K)$ double indexed sequence of rationals $(a^{(n)}_k)_k, N_n)$ from the corresponding to $(x_n)_n$ by $x_n = ((a^{(n)}_k)_k, N_n))$. This was achieved by setting $b_n := a^{(n)}_{N_n(n)}$.
Hence, we can give a first definition of Limit in MINLOG, by deriving $(b_n)_n$ from $(x_n)_n$ as a predicate-constant `RealLim`, using the computation rule:
`RealLim`$((x_n)_n, M)$:=$((x_n)seq_{(x_n)mod(n)}, p \mapsto max(M(p+1), p+2)) \in \mathbb{R}$
Hence, in MINLOG one writes:

```
(add-program-constant "RealLim" (py "(nat=>rea)=>(pos=>nat)=>rea"))
(add-computation-rules
 "RealLim xs M"
 "RealConstr([n](xs n)seq((xs n)mod(cNatPos n))) ([p]M(PosS p)max PosS(PosS p))")
```

This of course gives us the desired candidate of a Limit for x_n, as it directly coindcides with our natural definition. However, using `RealLim` in our MINLOG-proofs can lead to unwanted unfoldings, when we are normalizing the goal. This is just the same as it was with `GRec` and `NatToPos`[1].
After we have checked that `RealLim` is total, we then prove

`RLim`: $\forall (x_n)_n, M \exists x (x =$`RealLim`$(x_n)_n, M)$.

Since `RealLim` is total, this provides us with a witness `cRLim` to `RLim`, which has the same computation rules as `RealLim`. Of course this gives that

$\forall (x_n)_n, M$ (`cRLim`$((x_n)_n, M)$) $=^2$ `RealLim`$((x_n)_n, M)$ (`RLimExFree`).

[1]cf. "pos.scmänd the corresponding part of the script
[2]this "=" is Leibniz-Equality

With the command `(deanimate)`, we can now remove the computation rules `cRLim` again and receive a program constant `cRLim` resembling the candidate for the limit, which cannot be unfolded. However, if we do wish to unfold `cRLim` we can always use `RLimExFree`.

6 Step 2: cRLim is a Real (RealLimReal)

We will simply follow the procedure of the **2.Step** of the natural proof with MINLOG. Hence, we shall first prove the auxiliary claims `RealCompleteAux1`: "$(b_n)_n$ follows $(x_n)_n$" and `RealCompleteAux2`: "$(b_n)_n$ is a Cauchy-Sequence". Just then we're ready to prove **2.Step**(`RealLimReal`)

6.1 RealCompleteAux1

```
(set-goal "all ass,Ns,xs,bs(
 all n xs n eqd RealConstr(ass n)(Ns n) -> all n Real(xs n)
 -> all n bs n=ass n(Ns n(cNatPos n)) -> all n abs(bs n+ ~(xs n))<<=(1#2**n))")
```

Fixing variables `ass,Ns,xs,bs` and the assumptions on their properties, we obtain the assumptions

```
 ass  Ns  xs  bs  xsDef:all n xs n eqd RealConstr(ass n)(Ns n)
 Rxs:all n Real(xs n)  bsDef:all n bs n=ass n(Ns n(cNatPos n))  n
```

By the way it is used our goal, `n` has to be of type $\mathbb{N}$. By our Definitions of Cauchy Sequences, however, we can only get, that some absolute value is smaller than $\frac{1}{2^p}$ for some $p \in \mathbb{P}$. We therefore have to evaluate

```
(use "RealLeTrans" (pt "RealConstr([n0](1#2**cNatPos n))([p]Zero)"))
```

This yields the goals: `?^3:abs(bs n+ ~(xs n))<<=(1#2**cNatPos n)` and `?^4:(1#2**cNatPos n)<<=(1#2**n)`. Note that we will only discuss `?^3`, since proving `?^4` only invovles statements from `"nat.scm"`,`pos.scm"` and `"rat.scm"`.
By plugging in the Definions `xsDef` and `bsDef`, just as we did in the natural proof, we arrive at the goal:

```
?^6:abs(ass n(Ns n(cNatPos n))+ ~(RealConstr(ass n)(Ns n)))<<=(1#2**cNatPos n)
```

Now `ass n`, which is the sequence $(a_k^{(n)})_k$, is a Cauchy-sequence wiht modulus `Ns n`, since `xs`∈`Real`. We can therefore use `RatCauchyConvMod` and obtain the two goals

`?^7:Real(RealConstr(ass n)(Ns n)` and `?^8:Ns n(cNatPos n)<=Ns n(cNatPos n)`

Now, these two easily follow by assumption `Rxs` and `"Truth"`.

6.2 RealCompleteAux2

The next step in the natural proof of `RealLimReal` was to show that $(b_n)_N$ is a Cauchy-sequence. In the MINLOG program `librseq16.scm` this is done by proveing `RealCompleteAux2`

```
(set-goal "all ass,Ns,xs,M,bs,K(
 all n Real(xs n) ->
all p,n,m(M p<=n -> M p<=m -> abs(xs n+ ~(xs m))<<=(1#2**p))
-> all n xs n eqd RealConstr(ass n)(Ns n) -> all n bs n=ass n(Ns n(cNatPos n))
-> all p K p=M(PosS p)max PosS(PosS p) -> all p,n,m(K p<=n -> K p<=m -> abs(bs n+ ~(bs m))<<
```

We therefore fix the variables and take the assumptions below

```
 ass  Ns  xs  M  bs  K
 Rxs: all n Real(xs n) xsCs:all p,n,m(M p<=n -> M p<=m -> abs(xs n+ ~(xs m))<<=(1#2**p))
 xsDef:all n xs n eqd RealConstr(ass n)(Ns n) bsDef:all n bs n=ass n(Ns n(cNatPos n))
```

```
KDef:all p K p=M(PosS p)max PosS(PosS p)
p  n  m
nBd:K p<=n mBd:K p<=m
```

In view towards the inequality in our natural proof, one first asserts that `?^3:abs(bs n+ ~(bs m))<<=:RealPlus(1#2**n)(1#2**PosS p)+(1#2**m)`. Then, one also has to translate $|b_n - b_m|$, which are operations in $\mathbb{Q}$, into $\mathbb{R}$, in order to be able insert x_n and then use the triangle inequality. This is done by using

```
(use "RealLeTrans" (pt "abs(bs n+RealUMinus(bs m))")
```

on `?^3`. This splits `?^3` into two new goals `?^5` and `?^6`, where `?^5:abs(bs n+ ~(bs m))<<=abs(bs n+ ~(bs m)` (where the leftside is in $\mathbb{R}$ and the right one in $\mathbb{Q}$). This can easily be proved using simplifications. `?^6` then is the version of `?^3` with operations in $\mathbb{R}$.
For `?^6` we then bring x_n and use the triangle-inequality on the resulting term, using the comands:

```
(simpreal (pf "bs n+RealUMinus(bs m)===bs n+ ~(xs n)+(xs n+ RealUMinus(bs m))"))
(use "RealLeTrans" (pt "abs(bs n+ ~(xs n))+abs(xs n+ RealUMinus(bs m))"))
```

Note, that the asserted equality `bs n+RealUMinus(bs m)===bs n+ ~(xs n)+(xs n+ RealUMinus(bs m))` and the inequality `abs(bs n+ ~(xs n))+abs(xs n+ RealUMinus(bs m))` can easily be proved by using `RealPlusInsert`[3] and `RealLeAbsPlus`[4]. One then arrives at the goal

```
?^14:abs(bs n+ ~(xs n))+abs(xs n+ ~(bs m))<<=
     RealPlus(1#2**n)(1#2**PosS p)+(1#2**m)
```

Now, we consider `RealLeMonPlusTwo`, which states that

$$\forall x, y, z, z0\ (x <<= y \to z <<= z0 \to x + z <<= y + z0)$$

Using this Theorem, it suffices to show the two following goals, in order to obtain `?^14`.

```
?^21:abs(bs n+ ~(xs n))<<=(1#2**n)
?^22:abs(xs n+ ~(bs m))<<=RealPlus(1#2**PosS p)(1#2**m)
```

We instantly see that `?^21` exactly matches `RealCompleteAux1`, hence we can easily close this goal, since the definitions of `xs,bs` etc. are available as assumptions.
It remains to prove `?^22`. Because of `RealPlusInsert` and `RealLeAbsPlus`, we can insert `xs m` into `?^22` and then use triangle-inequality, just like it was done at `?^6`. Again, with `RealLeMonPlusTwo` we can split into the two goals:

```
?^32:abs(xs n+ ~(xs m))<<=(1#2**PosS p)
?^33:abs(xs m+ ~(bs m))<<=(1#2**m)
```

Now, `?^32` just corresponds to `xs` being a Cauchy-Sequence. Hence, we can simply use assumption `xsCS` and plug in the definition of `K` to get that `n,m`$\leq$`M(p+1)`.
For `?^33` can then switch `xs m` and `bs m`, using basic simplifications, so that we arrive at

```
?^50:abs(bs m+ ~(xs m))<<=(1#2**m)
```

This is exactly the statement of `RealCompleteAux1` again! Hence, we finish `?^50` off, using `RealCompleteAux1` and our assumed definitions.
The proof in MINLOG conintues with proveing all the necessary equalities and triangle-inequalities, which we have used so far, as well as statements about realness of some fractions. However, we will omitt a careful discussion of those, as we have already argued in favor of the (in-)equalities and the proofs of the latter can be put together using theorems about Realness only.

[3] stating that $\forall x, y, z(Realx \to Realy \to Realz \to x + z === x +\ y + (y + z))$
[4] stating that $\forall x, y(Realx \to Realy \to abs(x + y) <<= absx + absy)$

6.3 RealLimReal

We're finally ready to prove the main goal of the **2.Step**: `RealLimReal`, which is

```
(set-goal "all xs,M(RealCauchy xs M -> Real(cRLim xs M))")
```

By this means, we shall prove that, for every Cauchy-Sequence of proper Reals, the limit-candidate, obtained by diagonalizaion, is a proper Real again.
After fixing `xs,M`, we derive from the assumption `RC: RealCauchy xs M`, that `xs` is a sequence of proper Reals and `M` is a monotone function. This is done by using `(assert)` for these statements and proveing them with the eliminations of `RealCauchy` (cf. 3.2). Therefore, we now have available:

```
xs  M  RC:RealCauchy xs M  Rxs:all n Real(xs n)  MonM:Mon M
RCxsM:RCauchy xs M
```

We would also like to introduce the notation of our natural proof using `(ass,Ns)` for the sequence of rationals and `(bs,K)` for the diagonal sequence. This, again, is achieved by `(assert)`. For example to introduce `ass`, we do

```
(assert "exl ass all n ass n eqd(xs n)seq")
```

This comand gives the existence statement above as a new goal `^?16`. By the natural definition of `ass`, we know that this sequence is obtained from `xs`, by settiing `ass:=[n](xs n)seq`. Hence, we use the introduction `(intro 0 (pt "[n](xs n)seq"))` on `^?16` and reach the goal `?^17:all n ([n0](xs n0)seq)n eqd(xs n)seq`. This goal can of course be proved using the introduction-axiom for Leibniz-Equality `InitEqD`[5]. If we carry out this assertion-procedure for `Ns`[6],`bs` and `K` as well, we are provided with the following additional assumptions:

```
ass     assDef:all n ass n eqd(xs n)seq
Ns      NsDef:all n Ns n eqd(xs n)mod
xsChar:all n xs n eqd RealConstr(ass n)(Ns n)
bs      bsDef:all n bs n=ass n(Ns n(cNatPos n))
K       KDef:all p K p=M(PosS p)max PosS(PosS p)
```

Having introduced our usual definitions now, we can start with the actual proof, where our goal remains to be `?^55:Real(cRLim xs M)`.
By plugging in the Definition `RLimExFree` of `cRLim` we can simplify this to `?^56:Real(RealLim xs M)`.
By the Introduction axiom of `Real`, we have to show that `^?57:(RealLim xs M)seq` is a Cauchy-sequence and `^?58:(RealLim xs M)mod` is a montone Function. We will start with `^?57`.
By the introduction of `Cauchy`, proveing `^?57` is equivalent to proveing:

```
?^59:allnc p,n,m((RealLim xs M)mod p<=n -> (RealLim xs M)mod p<=m
-> abs((RealLim xs M)seq n+ ~((RealLim xs M)seq m))<=(1#2**p))
```

Hence, we frst `p,n,m` and assume `nBd:(RealLim xs M)mod p<=n` and `mBd:(RealLim xs M)mod p<=m`.
Now, by definition `(RealLim xs M)seq` resp. `(RealLim xs M)mod` has to be equal to `bs` resp `K`. Hence, we can simplify `^?57`, using the asserted equalities `assDef,NsDef,bsDef`,etc. and reformulate `^?57` as

```
?^68:abs(bs n+ ~(bs m))<<=(1#2**p)
```

This is exactly, what `RealCompleteAux2` states! Hence, we use `RealCompleteAux2` on this goal with our notation and reduce `?^68` to show `K p`≤`n` and `K p`≤`m`. For for these two goals, however, we simply need to plug in the `KDef` and use `nBd:(RealLim xs M)mod p<=n` and `mBd`. So, we're done with `?^57`.
It remains to prove `^?58:Mon((RealLim xs M)mod)`. Note, that by computation-rules of `RealLim`

[5] which reads as: $(\text{InitEqD})^+$: $\forall x\,(x\ eqd\ x)$

[6] where we also add the Characterisation xsChar, that (ass,Ns) is a defining Cauchy sequence for (xs,M)

we have `Mon((RealLim xs M)mod):=M(PosS p)max PosS(PosS p)`[7]. Now, this just involves basic reasoning for numbers in $\mathbb{P}$, such as that $p \leq p+2$ and using that `M` is mononte by `MonM`. Hence, we consider `?^58` to be closed.

7 3.Step: Convergence: RealCauchyConvMod, RealComplete

Recall that our main goal is to show `RealComplete:`, stating that any `RealCauchy`-sequence converges against the candidate `cRLim`. Before we are able to prove this theorem, however, we need to make one more auxiliary claim `RealCauchyConvMod`: "Any `RealCauchy`-sequence `(xs,M)` converges against the real given by `(bs,K)` with modulus `M`". Note, that whilst proveing `RealCauchyConvMod` we will follow the exact procedure of **3.Step** of our natural proof. `RealComplete`, however, will just consists of using `RealCauchyConvMod` on `cRLim` then.

7.1 RealCauchyConvMod

We shall prove `REalCauchyConvMod` now, which reads in MINLOG as:

```
(set-goal "all ass,Ns,xs,M,bs,K,x(
 all n xs n eqd RealConstr(ass n)(Ns n) -> RealCauchy xs M
 -> all n bs n=ass n(Ns n(cNatPos n)) -> all q K q=M(PosS q)max PosS(PosS q)
 -> x===RealConstr bs K -> all p,n(M p<=n -> abs(xs n+ ~x)<<=(1#2**p)))")
```

Fixing variables and assumption we'd have to derive `?^2:abs(xs n+ ~x)<<=(1#2**p)` from

```
ass  Ns  xs  M  bs  K  x
xsDef: all n xs n eqd RealConstr(ass n)(Ns n)
CxsM:RealCauchy xs M  bsDef:all n bs n=ass n(Ns n(cNatPos n))
KDef:all q K q=M(PosS q)max PosS(PosS q)  xEq:x===RealConstr bs K
p  n  nBd:M p<=n
```

Again, we can use the eliminations for `RealCauchy` and obtain the following valid, additional assumptions by `(assert)`:

```
Rxs:all n Real(xs n)    RCxsM:RCauchy xs M
```

Just as we did in the natural proof, we can reformulate `?^2`. Hence, by using `RealLeAllPlusToLe`[8], we see that, to obtain `?^2`, it suffices to show, that for every arbitrary `q` holds:

```
?^14:abs(xs n+ ~x)<<=RealPlus(1#2**p)(1#2**q)
```

Since the `?^2` and `?^14` for arbitrary `q` are clearly equivalent.
Recall, that in the natural proof, we now inserted a x_m into $|x_n - x|$ to then use the triangle-inequality, where we chose $m := max(M(p), K(p+1), p+1)$. We can do the same here! So we first define the m by the comand:

```
(defnc "m" "(M p)max(PosS q max K(PosS q))")
```

Then we insert `xs m` and use the triangle inequality, using similiar comand as we did in the proof of `RealCompleteAux2`. This gives the goal

```
?^25:abs(xs n+ ~(xs m))+abs(xs m+ ~x)<<=RealPlus(1#2**p)(1#2**q)
```

Also, we obtain a few side-goals with this procedure, again. But, as we saw in the `RealCompleteAux2`, these can easily be proved proved by using `RealPlusInsert`[9] and `RealLeAbsPlus`[10], so we'll omit

[7] this term just stand for the function $p \mapsto max(M(p+1), p+2)$
[8] which stands for: $\forall x, y(Realx- > Realy- > allpx <<= y + \frac{1}{2^p} - > x <<= y)$
[9] stating that $\forall x, y, z(Realx \to Realy \to Realz \to x + z === x + \ y + (y + z))$
[10] stating that $\forall x, y(Realx \to Realy \to abs(x+y) <<= absx + absy)$

them from our discussion. Following RealCompleteAux2, we can use RealLeMonPlusTwo, to split ?^25 into the two goals

```
?^28:abs(xs n+ ~(xs m))<<=(1#2**p)
?^29:abs(xs m+ ~x)<<=(1#2**q)
```

By careful observation of ?^28 one notices, that this is equal to claiming that xs is Cauchy sequence. We can hence use the assumption RCxsM, saying exactly that xs is a RCauchy-Sequence with modulus M. With nBd and mDef we can also prove that $n, m \geq M(p)$, which finishes the proof.

It remains to prove ?^29. Recall that we achieved this in the natural proof by inserting b_m, using the triangle equaltiy and then showing that $|b_m - x_m| \leq \frac{1}{2^{p+1}}$ by RealCompleteAux1 and $|b_m - x| \leq \frac{1}{2^{p+1}}$ by using RatCauchyConvMod.

The first thing to do in MINLOG is therefore to reformulate our claim to be

```
?^34:abs(xs m+ ~x)<<=RealPlus(1#2**PosS q)(1#2**PosS q)
```

Since, we do this using simplification on the equality $\frac{1}{2^p} = \frac{1}{2^{p+1}} + \frac{1}{2^{p+1}}$, we will have to prove this equality later (which is the goal ?^35. This however, will be a simple application of the theorem RatPlusHalfExpPosS[11]. Thus, it suffices to deal with ?^34.

Here, we insert bs m, use the triangle-inequality and split our claim, just like we did above. This produces the two goals

```
?^42:abs(xs m+ ~(bs m))<<=(1#2**PosS q)
?^43:abs(bs m+ ~x)<<=(1#2**PosS q)
```

Now, regarding ?^42, we can use RealCompleteAux1 to get that abs(bs m+ ~(xs m))<<=(1#2**m) ,after we have switched xs m and bs m in the statement. We can therefore close off ?^42, if we prove:

```
?^59:(1#2**m)<<=(1#2**PosS q)
```

Note, that we will omit a discussion, how this is proved, because it involves statements about Real and rational arithmetic and pluging in the definition of m. The interested reader might have a look at librseq16.scm

It remains to prove ?^43. If we plug in the definition of x: xEq now. ?^43 simplifies to

```
?^68:abs(bs m+ ~(RealConstr bs K))<<=(1#2**PosS q)
```

Since bs is a Cauchy-Sequence by RealCompleteAux2, we can now apply RatCauchyConvMod to obtain this goal and therefore finish the proof. □

7.2 RealComplete

To finalize our proof of the Completeness of Reals in MINLOG we need to show that the statement of RealCauchyConvMod also holds for our candidate for the limit: cRLim. Hence, we wish to prove:

```
(set-goal "all xs,M(RealCauchy xs M ->
 all p,n(M p<=n -> abs(xs n+ ~(cRLim xs M))<<=(1#2**p)))")
```

We can then obtain the usual notation for the natural proof as assumptions by makeing suitable assertions like we did above. Fixing the variables xs, M and later p,n and using the Eliminations for RealCauchy we obtain the assumptions:

```
 xs  M  RC:RealCauchy xs M   Rxs:all n Real(xs n)
 MonM:Mon M   RCxsM:RCauchy xs M   ass  assDef:all n ass n eqd(xs n)seq
 Ns  NsDef:all n Ns n eqd(xs n)mod   xsChar:all n xs n eqd RealConstr(ass n)(Ns n)
```

[11]which is precisely the statement that $forallp(\frac{1}{2^p} = \frac{1}{2^{p+1}} + \frac{1}{2^{p+1}}$.

```
 bs  bsDef:all n bs n=ass n(Ns n(cNatPos n))  K  KDef:all p K p=M(PosS p)max PosS(PosS p)
 p  n
```

Having taken those assumptions we still need to prove the goal

```
?^56:M p<=n -> abs(xs n+ ~(cRLim xs M))<<=(1#2**p)
```

As we know by `RealCauchyConvMod`, the statement `?^56` holds for any `x`, which satisfies `x eqd (bs, K)`. Thus, we use `RealCauchyConvMod` on `?^56` together with the asserted Definitions and obtain the goal:

```
?^61:cRLim xs M===RealConstr bs K
```

However, `:cRLim xs M` is Leibniz-equal to `RealLim xs M`, as we know by `RLimExFree`. Hence, we can replace `:cRLim xs M` by `RealLim xs M` in the goal `?^61`. Now, we'll try to prove the newly obtained goal, by unraveling the Definition of `RealLim xs M`. To be more precise, it suffices to show, that the defining sequences of `RealConstr bs K` and `RealLim xs M` coincide at some point and ongoing from that point. In MINLOG, we have proved this statement, called `RealSeqEqToEq`, in `rea.scm`. Using it on our goal, with chosen starting point of Coincidence: 0, gives:

```
?^63:Real(RealLim xs M)
?^64:Real(RealConstr bs K)
?^65:all n(Zero<=n -> (RealLim xs M)seq n==(RealConstr bs K)seq n)
```

Now, if replace `RealLim xs M` in `^?63` with `:cRLim xs M` again, we observe that the resulting goal exactly match bei b) bei der es the statement of `RealLimReal!`. Hence, using `RealLimReal` closes of `^?63`, since the assumption, that `xs M` is `RealCauchy`-sequence is availabel by `RC`.
We can also close off `^?64`, since we know that, given our notation in the assumptions above, `bs K` is a `Cauchy`-sequence by `RealCompleteAux2` with monotone modulus `K`.
It remains to show `?^65`. After fixing a variable `n1` for `n`, we can plug in the computation-rules with `(ng #t)` and obtain the goal:

```
?^94:(xs n1)seq((xs n1)mod(cNatPos n1))==bs n1
```

Using that `xs` was defined via `ass` by `xsChar`, we can further normalize this to the goal to

```
?^96:ass n1(Ns n1(cNatPos n1))==bs n1
```

This is of course true by the definition `bsDef`. □

8 Reformulation of RealComplete

Having proved that the `Reals` are complete, we might want to have a more natural way to speak about limits and convergence, than always having to deal with in-equalities and absolute values. We therefore propose two new predicates `RConvLim` and `RealConvLim`, each stateing that a given sequence `xs` converges (by the means above) agains an `x` with modulus `M`. Hence, we define

Definition 8.1
`RConvLim`$:=\mu_I(\forall (x_n)_n, x, M\ \ (\forall p, n\, M(p) \leq m \to |x_n - x| \leq \frac{1}{2}) \to I((x_n)_n, x, M))$

which is the version of general, possibly non-constructive reals. The next one is for constructive Reals then:

Definition 8.2
`RealConvLim`$:=\mu_I(\forall (x_n)_n, x, M\ \ (\forall n\,(x_n \in$`Real`$) \to x \in$`Real` $\to M \in$`Mon` $\to I((x_n)_n, x, M))$

We hence obtain the Introduction and Elimination-Axioms:

These are proved in MINLOG in the usual manner.
of course, we would like to verify that the predicates `RConvLim`, `RealConvLim` fit into the context of the previous theorems. More presicely, we'd like to prove that every `RealCauchy`-sequence has the property `RealConvLim` with the limit `cRLim`.

Theorem 8.3
`RealCompleteCor`*:* $\forall$`xs,x,M` *(*`RealCauchy(xs,M)`$\rightarrow$ `RealConvLim(xs, cRLim(xs,M), M`*))*

The prove of this theorem in MINLOG is rather simple. One only has to use the previosly proved statements: `RealCauchyToReals`, `RealLimReal`, `RealCauchyToMon` and last, but not least `RealComplete`.

9 Lookahead: Arithmetics of Limits