Infinite Series

Infinite series are handled in Minlog as a special type of sequence with modulo, analogous to how series are treated in classical analysis (minus the modulo). This yields the following definition:

Definition (RSerConvLim)
A series $\sum_{k=0}^{\infty} x_k$ is said to converge to x with modulo M iff
$\forall p, n : (M(p) \leq n \to |(\sum_{k=0}^{n} x_k) - x| \leq \frac{1}{2^p})$ [1]

Note that MINLOG introduces this and the coming definitions twice, although in one case the series is explicitly required to have only real parts and the modulo has to be increasing. For the sake of simplicity, I shall not deal with these slightly different definitions, as proofs in the second case are also usually just reduced to the first.

This second definition is the analogue to Cauchyness for series:
Definition (RSerConv)
A series is $\sum_{k=0}^{\infty} x_k$ said to be Cauchy with modulo M iff
$\forall p, n, m : (M(p) \leq n \to |\sum_{k=n}^{n+m-1} x_k| \leq \frac{1}{2^p})$ [2]

This now lets us prove out first theorem:
Theorem (RSerAbsConvToConv)
$\sum_{k=0}^{\infty} |x_k|$ is Cauchy with modulo $M \Rightarrow \sum_{k=0}^{\infty} x_k$ is Cauchy with modulo M

[1] The types of the variables should be clear from the context. I will only specify, where confusion might arise.

[2] If the use of the natural number m seems odd in this definition, it is useful to remember the way sums are defined in MINLOG

The proof of this theorem mainly uses the triangle inequality for finite sums (RealLeAbsSum), which will not be shown in this article.
It is also important to note here, that in MINLOG we, at different stages in this article, have to prove a number of small theorems about index shifts for series, and being able to split sums up into parts, etc. While conceptually very easy to understand, the proofs of these theorem can be quite tricky to actually complete in MINLOG. I will however use these theorems when necessary without much comment.

Let us now move on to a different topic, namely the binomial theorem:
<u>**Theorem**</u> (RealBinom)
$\forall x, y \in \mathbb{R}\, \forall n \in \mathbb{N} : (x+y)^n = \sum_{m=0}^{n} x^{n-m} y^m \binom{n}{m}$ [3]

Proof:
We prove the theorem by induction. As the base case is clear, let's move on to the successor case. We have:
$(x+y)^{n+1} = (x+y)(x+y)^n = x(x+y)^n + y(x+y)^n$
$\overset{IH}{=} x \sum_{m=0}^{n} x^{n-m} y^m \binom{n}{m} + y \sum_{m=0}^{n} x^{n-m} y^m \binom{n}{m}$
$= \sum_{m=0}^{n} x^{n+1-m} y^m \binom{n}{m} + \sum_{m=0}^{n} x^{n-m} y^{m+1} \binom{n}{m}$
$= x^{n+1} + \sum_{m=1}^{n} x^{n+1-m} y^m \binom{n}{m} + \sum_{m=0}^{n} x^{n-m} y^{m+1} \binom{n}{m}$
$= x^{n+1} + \sum_{m=1}^{n} x^{n+1-m} y^m \binom{n}{m} + \sum_{m=1}^{n+1} x^{n+1-m} y^m \binom{n}{m-1}$
$= x^{n+1} + \sum_{m=1}^{n+1} x^{n+1-m} y^m \binom{n}{m} + \sum_{m=1}^{n+1} x^{n+1-m} y^m \binom{n}{m-1}$ (as $\binom{n}{n+1} := 0$)
$= x^{n+1} + \sum_{m=1}^{n+1} x^{n+1-m} y^m (\binom{n}{m} + \binom{n}{m-1})$

[3]Note here again that equality technically should be read as ===, which is simply an equivalence relation with "nice" properties. For the sake of these proofs however, we need no Leibniz-equality, so I shall use the usual symbol.

$= x^{n+1} + \sum_{m=1}^{n+1} x^{n+1-m} y^m \binom{n+1}{m}$ (Pascal's identity)
$= \sum_{m=0}^{n+1} x^{n+1-m} y^m \binom{n+1}{m}$
This was what we wanted to show, so by induction we are done.

Now we shall concern ourselves with one of the classic convergence tests for series, namely the comparison and quotient test. Here our main goal is to construct an explicit Cauchy modulo. The approach here is to show the theorems for basic cases first (i.e. the sequences is dominated from the beginning) and then move on to more complex cases.

Theorem (RComparisonTest)
For sequences $(x_k)_k, (y_k)_k$ with $\forall k : 0 \leq x_k \leq y_k$, we have that
$\sum_{k=0}^{\infty} y_k$ is Cauchy with modulo $M \Rightarrow \sum_{k=0}^{\infty} x_k$ is Cauchy with modulo M

A large portion of the proof in MINLOG is dedicated to showing
$\sum_{k=n}^{n+m-1} |x_k| = \sum_{k=n}^{n+m-1} x_k \geq 0$.
Once this is shown, we simply have
$\forall p, n, m : (M(p) \leq n \to \sum_{k=n}^{n+m-1} |x_k| = \sum_{k=n}^{n+m-1} x_k \leq \sum_{k=n}^{n+m-1} y_k \leq$
$|\sum_{k=n}^{n+m-1} y_k| \leq \frac{1}{2^p})$

Let us now generalize a bit:
Lemma (RComparisonTextMax)
For sequences $(x_k)_k, (y_k)_k$ and a natural number l with $\forall k : 0 \leq x_k$ and $\forall k \geq l : x_k \leq y_k$, we have that
$\sum_{k=0}^{\infty} y_k$ is Cauchy with modulo $M \Rightarrow \sum_{k=0}^{\infty} x_k$ is Cauchy with modulo $N(p) = max(M(p), l)$

The main gist of this generalization is the change in modulo. Since $N(p) \geq l$ for all p, we have that $N(p) \leq n \to l \leq n$ and so we can apply RComparisonTest.

Before we can move on to the comparison test, we need to show a few Lemmas. I will not go through the proof of these, but state some of them here.

Lemma (RCauchyTimesConstR)
If $\sum_{k=0}^{\infty} x_k$ is Cauchy with modulo M and y is a real numbers with $y <= 2^q$ for a natural number q, then $\sum_{k=0}^{\infty} y x_k$ is Cauchy with modulo $N(p) = M(p+q)$

Lemma (RSerConvShiftUp)
If $\sum_{k=0}^{\infty} x_k$ is Cauchy with modulo M and $l \in \mathbb{N}$, then $\sum_{k=0}^{\infty} x_{k+l}$ is Cauchy with modulo $N(p) = M(p) - l$

None of these Lemmas should be too surprising, and some more similar Lemmas shall also be used throughout this article.

Before we can finally prove the ratio test, we need one more final theorem:
Theorem (RCauchyExpToRSerConvExp)
If $0 \leq x$ and $1 - x \in_p \mathbb{R}^+$, and $(x^n)_{n \in \mathbb{N}}$ is a Cauchy-sequence with modulo M (it will be a Null-sequence, but the modulo is explicitly important here), then $\sum_{k=0}^{\infty} x^k$ is Cauchy with modulo $N(p_0) = M(p_0 + p + 1)$

Proof: For $M(p_0 + p + 1) \leq n$ and $m \in \mathbb{N}$ we have
$|\sum_{k=n}^{n+m-1} x^k| = |(\sum_{k=0}^{n+m-1} x^k) - (\sum_{k=0}^{n-1} x^k)| \overset{GeomSumEq}{=} |\frac{1-x^{n+m}}{1-x} - \frac{1-x^n}{1-x}|$

$= |\frac{x^n - x^{n+m}}{1-x}| = \frac{|x^{n+m} - x^n|}{|1-x|} \leq \frac{\frac{1}{2^{p_0+p+1}}}{|1-x|} \overset{RealPosChar}{\leq} \frac{1}{2^{p_0+p+1}} 2^{p+1} = \frac{1}{2^{p_0}}$ [4]

Now we are finally ready to prove the constructive version of the ratio-test:

Theorem (RealRatioTestZero)

Let $(x_n)_n$ be a real sequence, $|x_0| \leq 2^q$ for some $q \in \mathbb{N}$ and $0 \leq z \in \mathbb{R}$ with $1 - z \in_p \mathbb{R}$, $(z^n)_{n \in \mathbb{N}}$ Cauchy with modulo M and $\forall n \in \mathbb{N} : |x_{n+1}| \leq z|x_n|$, then $\sum_{k=0}^{\infty} |x_k|$ is Cauchy with modulo $N(p_0) = M(q + p_0 + p + 1)$

Proof: We first show by induction, that $\forall k \in \mathbb{N} : |x_k| \leq z^k |x_0|$. Then we have for $M(q + p_0 + p + 1) \leq n$ and $m \in \mathbb{N}$ that

$\sum_{k=n}^{n+m+1} |x_k| \leq \sum_{k=n}^{n+m+1} z^k |x_0| = |x_0| \sum_{k=n}^{n+m+1} z^k \leq^{5} |x_0| \frac{1}{2^{q+p_0}} \leq \frac{2^q}{2^{q+p_0}} = \frac{1}{2^{p_0}}$

Applying the same trick as in the comparison-test case, we get:

Theorem (RealRatioTestMax)

Let $(x_n)_n$ be a real sequence, $m \in \mathbb{N}, |x_m| \leq 2^q$ for some $q \in \mathbb{N}$ and $0 \leq z \in \mathbb{R}$ with $1 - z \in_p \mathbb{R}$, $(z^n)_{n \in \mathbb{N}}$ Cauchy with modulo M and $\forall n \geq m : |x_{n+1}| \leq z|x_n|$, then $\sum_{k=0}^{\infty} |x_k|$ is Cauchy with modulo $N(p_0) = max(M(q + p_0 + p + 1), m)$

Pair-Encoding

Before we can move on to prove the Cauchy-Product theorem, we need to have a clean way of writing the product of two sums as one sum over pairs. We do this by "filling out the square". For this, we're going to use a natural encoding of pairs that arises in this context.

[4] The reason for the somewhat odd disappearance of the +1 in RealPosChar lies in a technical detail in the definition of $\in_p \mathbb{R}$

[5] See the modulo of the geometric series in RCauchyExpToRSerConvExp

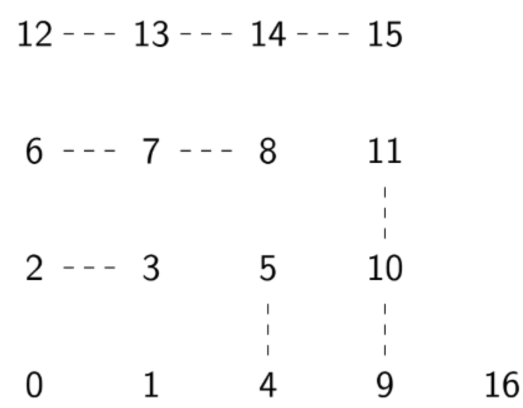

Figure 1: Pair-encoding

Here, we show the encoding $code(i,j)$ where i and j are numbered (from 0) on the x-axis and y-axis respectively.
From this idea, we can already extract a possible formula for our encoding, namely

$$code(i,j) = \left\{ \begin{array}{ll} i^2 + j, & \text{for } j < i \\ j^2 + j + i, & \text{for } i \leq j \end{array} \right\}$$

To motivate this formula: Note that $code(i,0) = i^2$ and $code(0,j) = j^2 + j$. Then the rest of the formula follows by noting $code(i,j) = code(i,0) + j$ for $j < i$ and $code(i,j) = code(0,j) + i$ for $i \leq j$.
It should be clear for why this encoding is called ”filling out the square”.
Now, as this encoding is supposed to be a bijection, $code : \mathbb{N}^2 \to \mathbb{N}$ needs to have an inverse(s) $(codeL, codeR) : \mathbb{N} \to \mathbb{N}^2$ such that
$\forall n \in \mathbb{N} : code(codeL(n), codeR(n)) = n$ and
$\forall i, j \in \mathbb{N} : codeL(code(i,j)) = i \wedge codeR(code(i,j)) = j$.
Let us try to construct these: Before we can do this, lets define a function that returns the ”natural root” of a number n, i.e. the largest natural

number k such that $k^2 \leq n$, or, in term of our graph, the (size of the) square that n lies in in our encoding. We define the function Rt recursively:

$Rt(0) = 0$

$$Rt(n+1) = \left\{ \begin{array}{ll} Rt(n), & \text{if } n < Rt(n)^2 + 2Rt(n) \\ j^2 + j + i, & \text{else} \end{array} \right\}$$

The reader is invited to ponder this definition until convinced by it, we suggest looking at the above graph..

Now using the Rt function, we can finally define our function $CodeL$ and $CodeR$. We have:

$$codeL(n) = \left\{ \begin{array}{ll} Rt(n), & \text{for } n < Rt(n)^2 + Rt(n) \\ n - (Rt(n)^2 + Rt(n)), & \text{else} \end{array} \right\}$$

$$codeR(n) = \left\{ \begin{array}{ll} n - Rt(n)^2, & \text{for } n < Rt(n)^2 + Rt(n) \\ Rt(n), & \text{else} \end{array} \right\}$$

Again, please ponder this definition until it becomes clear.

It comes with some difficulty to show that these functions actually do (in their pair) constitute an inverse, but I hope it is intuitively somewhat clear. In the following I shall write cL and cR for $codeL$ and $codeR$ respectively.

Before me move on, lets develop some useful intuitions.

First of all. we have that the n-th square (i.e. with sidelength n) is given by $\{k \in \mathbb{N} : max(cL(k), cR(k)) \leq n\}$, because $max(cL(k), cR(k)) = Rt(k)$. Note that the $n-1$-th square is also given by $\{k \in \mathbb{N} : k \leq n^2 - 1\}$, which yields the following equality

$\{k \in \mathbb{N} : k \leq n^2 - 1\} \ = \{k \in \mathbb{N} : max(cL(k), cR(k)) < n\}$.

Thus, the $n-1$-th square minus the $m-1$-th square is given by
$\{k \in \mathbb{N} : m \leq max(cL(k), cR(k)) \wedge k \leq n^2 - 1\}$.
Furthermore, the upper triangle in the $n-1$-th square (including the diagonal) is given by $\{k \in \mathbb{N} : n \leq cL(k) + cR(k) \wedge k \leq n^2 - 1\}$, the lower triangle (excluding the diagonal) is thus given by
$\{k \in \mathbb{N} : cL(k) + cR(k) < n \wedge k \leq n^2 - 1\}$.

Cauchy-Product

Now we finally have all the tools to prove the Cauchy-Product-Theorem. Note that in the following section, we revert back to viewing series as sequences, and talking about convergence of sequences. This is not for any strictly mathematical reasons, all these proofs could be carried out using only talk of convergence/Cauchyness of series. However, as we shall see, it is useful to let the sum only go up to $n-1$, which would change the modulo of convergence by 1. So instead of constantly having to add/subtract 1, for simplicity, we talk of series as sequences. First we shall prove some equalities that demonstrate the usefulness of our chosen pair-encoding.

Theorem (RealSumTimes)
$(\sum_{k=0}^{n-1} x_k)(\sum_{k=0}^{n-1} y_k) = \sum_{k=0}^{n^2-1} x_{cL(k)} y_{cR(k)}$

Proof: We prove the theorem by induction over n.
Base case ($n = 0$): Left-side $= 0 =$ Right-side
Induction step:
$(\sum_{k=0}^{n} x_k)(\sum_{k=0}^{n} y_k)$
$= x_n y_n + x_n \sum_{k=0}^{n-1} y_k + y_n \sum_{k=0}^{n-1} x_k + (\sum_{k=0}^{n-1} x_k)(\sum_{k=0}^{n-1} y_k)$

$\overset{IH}{=} x_n y_n + x_n \sum_{k=0}^{n-1} y_k + y_n \sum_{k=0}^{n-1} x_k + \sum_{k=0}^{n^2-1} x_{cL(k)} y_{cR(k)}$ (1)

We also have:

$\sum_{k=n^2}^{n^2+2n} x_{cL(k)} y_{cR(k)}$

$= \sum_{k=n^2}^{n^2+n-1} x_{cL(k)} y_{cR(k)} + \sum_{k=n^2+n}^{n^2+2n} x_{cL(k)} y_{cR(k)}$

$\overset{Def.\ von\ cL,cR}{=} \sum_{k=n^2}^{n^2+n-1} x_n y_{k-n^2} + \sum_{k=n^2+n}^{n^2+2n} x_{k-(n^2+n)} y_n$

$= x_n \sum_{k=0}^{n-1} y_k + y_n \sum_{k=0}^{n} x_k$ (Index-Shift)

$= x_n y_n + x_n \sum_{k=0}^{n-1} y_k + y_n \sum_{k=0}^{n-1} x_k$

It follows, that

$(1) = \sum_{k=0}^{n^2-1} x_{cL(k)} y_{cR(k)} + \sum_{k=n^2}^{n^2+2n} x_{cL(k)} y_{cR(k)} = \sum_{k=0}^{(n+1)^2-1} x_{cL(k)} y_{cR(k)}$,

which is exactly what we wanted to show.

Now we can show an essential inequality, namely:

Theorem (RealSquareMinusLowTriangEqUp, RealUpperTriangLeMinusSquare, RealSumMinusSquare)

For $2m \leq n$ we have

$\sum_{k=0}^{n^2-1} |x_{cL(k)}||y_{cR(k)}| - \sum_{cL(k)+cR(k)<n}^{n^2-1} |x_{cL(k)}||y_{cR(k)}|$

$= \sum_{n \leq cL(k)+cR(k)}^{n^2-1} |x_{cL(k)}||y_{cR(k)}|$

$\leq \sum_{m \leq max(cL(k),cR(k))}^{n^2-1} |x_{cL(k)}||y_{cR(k)}|$

$= \sum_{k=0}^{n^2-1} |x_{cL(k)}||y_{cR(k)}| - \sum_{k=0}^{m^2-1} |x_{cL(k)}||y_{cR(k)}|$

Proof idea:

The first equality should be clear. For the inequality, observe that $2m \leq n \leq cL(k)+cR(k) \Rightarrow 2m \leq cL(k)+cR(k) \Rightarrow m \leq max(cL(k), cR(k))$, so the inequality is just the result of summing over a larger domain. The last equality follows from the above note, namely that the $n-1$-th square minus the $m-1$-th square is given by

$\{k \in \mathbb{N} : m \leq max(cL(k), cR(k)) \wedge k \leq n^2 - 1\}$.

We now just need some theorems regarding the modulo of convergence of series. These are direct applications of convergence statements for sequences, however if the reader is not familiar with those, they also make for good exercises.

<u>Theorems</u>

1. Let $(\sum_{k=0}^{n-1} |x_k|)_{n\in\mathbb{N}}$, $(\sum_{k=0}^{n-1} |y_k|)_{n\in\mathbb{N}}$ be Cauchy (as sequences) with modulo M, N respectively, and $\sum_{k=0}^{\infty} |x_k| \leq 2^p$, $\sum_{k=0}^{\infty} |y_k| \leq 2^q$ for some $p, q \in \mathbb{N}$.
   Then $((\sum_{k=0}^{n-1} x_k)(\sum_{k=0}^{n-1} y_k))_{n\in\mathbb{N}}$ is Cauchy with modulo
   $K(r) = max(M(p + r + 1), N(q + r + 1))$
   (RealUpperTriangLimZeroAux)

2. Let $(\sum_{k=0}^{n-1} x_k)_{n\in\mathbb{N}}$, $(\sum_{k=0}^{\infty} y_k)_{n\in\mathbb{N}}$ converge to $x, y \in \mathbb{R}$ respectively with modulo M, N (again, as sequences). Furthermore, let $\forall n \in \mathbb{N} : |\sum_{k=0}^{n-1} x_k| \leq 2^p$, $|\sum_{k=0}^{n-1} y_k| \leq 2^q$ for some $p, q \in \mathbb{N}$. Then $\sum_{k=0}^{n^2-1} x_{cL(k)} y_{cR(k)} = (\sum_{k=0}^{n-1} x_k)(\sum_{k=0}^{n-1} y_k)_{n\in\mathbb{N}}$ converges to xy with modulo $K(r) := max(N(p + r + 1), M(q + r + 1))$
   (RealConvLimZStar)

<u>Proof:</u> (1.) follows from RealCauchyBdTimes, (2.) from RealConvLimTimes.

Theorem (RealCauchyProdLim)

Let $(\sum_{k=0}^{n-1} x_k)_{n\in\mathbb{N}}$, $(\sum_{k=0}^{n-1} y_k)_{n\in\mathbb{N}}$ converge to $x, y \in \mathbb{R}$ respectively with modulo M, N. Furthermore, let $\forall n \in \mathbb{N} : |\sum_{k=0}^{n} x_k| \leq 2^p$, $|\sum_{k=0}^{n} y_k| \leq 2^q$ for some $p, q \in \mathbb{N}$. Define $K(r) := max(N(p+r+1), M(q+r+1))$. Now assume $(\sum_{k=0}^{n-1} |x_k|)_{n\in\mathbb{N}}$, $(\sum_{k=0}^{n-1} |y_k|)_{n\in\mathbb{N}}$ are Cauchy (as sequences) with modulo M_0, N_0 respectively and $\sum_{k=0}^{\infty} |x_k| \leq 2^{p_0}$, $\sum_{k=0}^{\infty} |y_k| \leq 2^{q_0}$ for some $p_0, q_0 \in \mathbb{N}$. Define $K_0(r) := max(M_0(p_0+r+1), N_0(q_0+r+1))$.
Then $(\sum_{cL(k)+cR(k)<n}^{n^2-1} x_{cL(k)} y_{cR(k)})_{n\in\mathbb{N}}$ converges to xy with modulo
$Q(r) := max(2K_0(r+1), K(r+1))$

Proof: Let $max(2K_0(r+1), K(r+1)) \leq n$. Then
$|\sum_{cL(k)+cR(k)<n}^{n^2-1} x_{cL(k)} y_{cR(k)} - xy|$
$\leq |\sum_{cL(k)+cR(k)<n}^{n^2-1} x_{cL(k)} y_{cR(k)} - \sum_{k=0}^{n^2-1} x_{cL(k)} y_{cR(k)}| + |\sum_{k=0}^{n^2-1} x_{cL(k)} y_{cR(k)} - xy|$
$= |\sum_{n\leq cL(k)+cR(k)}^{n^2-1} x_{cL(k)} y_{cR(k)}| + |\sum_{k=0}^{n^2-1} x_{cL(k)} y_{cR(k)} - xy|$
$= |\sum_{n\leq cL(k)+cR(k)}^{n^2-1} x_{cL(k)} y_{cR(k)}| + |(\sum_{k=0}^{n-1} x_k)(\sum_{k=0}^{n-1} y_k) - xy|$
(RealSumTimes)
Now since $K(r+1) \leq n$, we have $|(\sum_{k=0}^{n-1} x_k)(\sum_{k=0}^{n-1} y_k) - xy| \leq \frac{1}{2^{r+1}}$ by RealConvLimZStar and thus:
$\leq |\sum_{n\leq cL(k)+cR(k)}^{n^2-1} x_{cL(k)} y_{cR(k)}| + \frac{1}{2^{r+1}}$
$\leq \sum_{n\leq cL(k)+cR(k)}^{n^2-1} |x_{cL(k)}||y_{cR(k)}| + \frac{1}{2^{r+1}}$
Now as we also have $2K_0(r+1) \leq n$ we can apply RealUpperTriangLeMinusSquare (to $m = K_0(r+1)$), to get:
$\leq \sum_{K_0(r+1)\leq max(cL(k),cR(k))}^{n^2-1} |x_{cL(k)}||y_{cR(k)}| + \frac{1}{2^{r+1}}$
$= \sum_{k=0}^{n^2-1} |x_{cL(k)}||y_{cR(k)}| - \sum_{k=0}^{K_0(r+1)^2-1} |x_{cL(k)}||y_{cR(k)}| + \frac{1}{2^{r+1}}$
(RealSumMinusSquare)
$= (\sum_{k=0}^{n-1} |x_k|)(\sum_{k=0}^{n-1} |y_k|) - (\sum_{k=0}^{K_0(r+1)-1} |x_k|)(\sum_{k=0}^{K_0(r+1)-1} |y_k|) + \frac{1}{2^{r+1}}$

(RealSumTimes)

$= |(\sum_{k=0}^{n-1} |x_k|)(\sum_{k=0}^{n-1} |y_k|) - (\sum_{k=0}^{K_0(r+1)-1} |x_k|)(\sum_{k=0}^{K_0(r+1)-1} |y_k|)\,| + \frac{1}{2^{r+1}}$

Now finally, using RealUpperTriangLimZeroAux, we get the last inequality:

$\leq \frac{1}{2^{r+1}} + \frac{1}{2^{r+1}}$

$= \frac{1}{2^r}$,

which is exactly what needed to be shown.