Constructive aspects of Riemann's permutation theorem for series

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Abstract

The notions of permutable and weak-permutable convergence of a series $\sum_{n=1}^{\infty} a_n$ of real numbers are introduced. Classically, these two notions are equivalent, and, by Riemann's two main theorems on the convergence of series, a convergent series is permutably convergent if and only if it is absolutely convergent. Working within Bishop-style constructive mathematics, we prove that Ishihara's principle BD-$\mathbf{N}$ implies that every permutably convergent series is absolutely convergent. Since there are models of constructive mathematics in which the Riemann permutation theorem for series holds but BD-$\mathbf{N}$ does not, the best we can hope for as a partial converse to our first theorem is that the absolute convergence of series with a permutability property classically equivalent to that of Riemann implies BD-$\mathbf{N}$. We show that this is the case when the property is weak-permutable convergence.

1 Introduction

This paper follows on from [3], in which the first two authors gave proofs, within the framework of Bishop-style constructive mathematics (**BISH**),[1] of the two famous series theorems of Riemann [18]:[2]

RST$_1$ *If a series $\sum a_n$ of real numbers is absolutely convergent, then for each permutation σ of the set $\mathbb{N}^+$ of positive integers, the series $\sum a_{\sigma(n)}$ converges to the same sum as $\sum a_n$.*

RST$_2$ *If a series $\sum a_n$ of real numbers is conditionally convergent, then for each real number x there exists a permutation σ of $\mathbb{N}^+$ such that $\sum a_{\sigma(n)}$ converges to x.*

[1] Roughly, **BISH** is mathematics using intuitionistic logic, a related set theory such as constructive ZF [1] or constructive Morse set theory [2], and dependent choice. For more on **BISH**, see [4, 5, 9, 10].

[2] We use shorthand like $\sum a_n$ and $\sum a_{\sigma(n)}$ for series when it is clear what the index of summation is.

It is not hard to extend the conclusion of **RST**$_2$ to what we call its *full, extended version*, which includes the existence of permutations of the series $\sum a_n$ that diverge to ∞ and to $-\infty$. In consequence, a simple *reductio ad absurdum* argument proves classically that if a real series $\sum a_n$ is **permutably convergent**—that is, every permutation of $\sum a_n$ converges in $\mathbb{R}$—then it is absolutely convergent. An intuitionistic proof of this last result was provided by Troelstra [20, pages 95ff.], using Brouwer's continuity principle for choice sequences. That result actually has one serious intuitionistic application: Spitters [19, pages 2101–2] uses it to give an intuitionistic proof of the characterisation of normal linear functionals on the space of bounded operators on a Hilbert space; he also asks whether there is a proof of the Riemann-Troelstra result within **BISH** alone. In Section 3 below, we give a proof, within **BISH** *supplemented by the constructive-foundationally important principle* BD-$\mathbb{N}$, that permutable convergence implies absolute convergence. While this proof steps outside unadorned **BISH**, it is valid in both intuitionistic and constructive recursive mathematics, in which BD-$\mathbb{N}$ is derivable; see [12, 13, 14].

This raises the question: over **BISH**, does the absolute convergence of every permutably convergent series imply BD-$\mathbb{N}$? Thanks to Diener and Lubarsky [11], we now know that the answer is negative. This raises another question: is there a proposition that is classically equivalent to, and constructively cognate with, the absolute convergence of all permutably convergent series and that, added to **BISH**, implies BD-$\mathbb{N}$? In order to answer this question affirmatively, we introduce in Section 2 the notion of *weak-permutable convergence* and then derive some of its fundamental properties, including its classical equivalence to permutable convergence. In Section 4 we show that the absolute convergence of weak-permutably convergent series implies BD-$\mathbb{N}$. Thus, in **BISH**, the statement

> every weak-permutably convergent series is absolutely convergent

implies BD-$\mathbb{N}$, which in turn implies that

> every permutably convergent series is absolutely convergent.

In view of the Diener-Lubarsky results in [11], the latter of these implications cannot be reversed.

2 Weak-permutably convergent series in BISH

By a **bracketing** of a real series $\sum a_n$ we mean a pair comprising

- a strictly increasing mapping $f : \mathbb{N}^+ \to \mathbb{N}^+$ with $f(1) = 1$, and
- the sequence $\mathbf{b}$ defined by

$$b_k \equiv \sum_{i=f(k)}^{f(k+1)-1} a_i \quad (k \geqslant 1).$$

By abuse of language, we also refer to the series $\sum b_k$ as a bracketing of $\sum a_n$.

We say that $\sum a_n$ is **weak-permutably convergent** if it is convergent and if for each permutation σ of $\mathbb{N}^+$ there exists a convergent bracketing of $\sum a_{\sigma(n)}$. Clearly, permutable convergence implies weak-permutable convergence. As we shall see in this section, the converse holds classically; later we shall show that it does not hold constructively. As a first step towards this, we have:

Proposition 1 *Let $\sum a_n$ be a weak-permutably convergent series of real numbers, with sum s, and let σ be a permutation of $\mathbb{N}^+$. Then every convergent bracketing of $\sum a_{\sigma(n)}$ converges to s.*

The proof of this proposition will depend on some lemmas.[3]

Lemma 2 *Let $\sum a_n$ be a convergent series of real numbers, with sum s, and let σ be a permutation of $\mathbb{N}^+$. If there exists a bracketing $(f, \mathbf{b})$ of $\sum a_{\sigma(n)}$ that converges to a sum $t \neq s$, then there exist a permutation τ of $\mathbb{N}^+$ and a strictly increasing sequence $(k_i)_{i \geqslant 1}$ of positive integers such*

$$\left| \sum_{n=f(k_i)}^{f(k_{i+1})-1} a_{\tau(n)} \right| \geq \frac{1}{3} |s-t| \qquad (i \in \mathbb{N}^+). \tag{1}$$

Proof. Consider, to illustrate, the case where $s < t$. For convenience, let $\varepsilon = \frac{1}{3}(t-s)$. Pick k_1 such that $\left|\sum_{n=j}^{k} a_n\right| \leq \varepsilon$ and $\left|\sum_{n=j}^{k} b_n\right| \leq \varepsilon$ whenever $k > j \geq f(k_1)$, and let $\tau(n) = \sigma(n)$ for $1 \leq n < f(k_1)$. Then

$$\begin{aligned} \sum_{n=1}^{f(k_1)-1} a_{\tau(n)} &= \sum_{n=1}^{f(k_1)-1} a_{\sigma(n)} = \sum_{j=1}^{k_1-1} \sum_{n=f(j)}^{f(j+1)-1} a_{\sigma(n)} = \sum_{j=1}^{k_1-1} b_j \\ &= \sum_{j=1}^{\infty} b_j - \sum_{j=k_1}^{\infty} b_j \geq t - \left| \sum_{j=k_1}^{\infty} b_j \right| \geq t - \varepsilon. \end{aligned}$$

Next, pick $k_2 > k_1$ such that

$$\{\tau(1), \ldots, \tau(f(k_1)-1)\} = \{\sigma(1), \ldots, \sigma(f(k_1)-1)\} \subset \{1, \ldots, f(k_2)-1\}.$$

There are exactly $f(k_2) - f(k_1)$ values of m in the interval $[1, f(k_2)-1] \cap \mathbb{N}$ such that $m \notin \{\sigma(1), \ldots, \sigma(f(k_1)-1)\}$. Set $\tau(f(k_1))$ equal to the smallest such m, $\tau(f(k_1)+1)$ equal to the next smallest, and so on. Then

$$\{\tau(1), \ldots, \tau(f(k_2)-1)\} = \{1, \ldots, f(k_2)-1\},$$

so

$$\sum_{n=1}^{f(k_2)-1} a_{\tau(n)} = \sum_{n=1}^{f(k_2)-1} a_n = s - \sum_{f(k_2)}^{\infty} a_n \leq s + \left| \sum_{f(k_2)}^{\infty} a_n \right| \leq s + \varepsilon.$$

[3] Following Bishop [4], when we write $s \neq t$ we mean that $|s-t| > 0$.

Note that if $f(k_1) \leq n < f(k_2) - 1$, then $\tau(n) = \sigma(k)$ for some $k \geq f(k_1)$. Now pick $k_3 > k_2$ such that

$$\{\tau(1), \ldots, \tau(f(k_2) - 1)\} \subset \{\sigma(1), \ldots, \sigma(f(k_3) - 1)\}.$$

There are exactly $f(k_3) - f(k_2)$ values of m in $[1, f(k_3) - 1] \cap \mathbb{N}$ such that $\sigma(m) \notin \{\tau(1), \ldots, \tau(f(k_2) - 1)\}$. Set $\tau(f(k_2))$ equal to $\sigma(m)$ for the smallest such m, $\tau(f(k_2) + 1)$ equal $\sigma(m)$ for the next smallest m, and so on. Then

$$\{\tau(1), \ldots, \tau(f(k_3) - 1)\} = \{\sigma(1), \ldots, \sigma(f(k_3) - 1)\},$$

so

$$\sum_{n=1}^{f(k_3)-1} a_{\tau(n)} = \sum_{n=1}^{f(k_3)-1} a_{\sigma(n)} = \sum_{j=1}^{k_3-1} \sum_{n=f(j)}^{f(j+1)-1} a_{\sigma(n)} = \sum_{j=1}^{k_3-1} b_j$$
$$= \sum_{j=1}^{\infty} b_j - \sum_{j=k_3}^{\infty} b_j \geq t - \left| \sum_{j=k_3}^{\infty} b_j \right| \geq t - \varepsilon.$$

Now pick $k_4 > k_3$ such that

$$\{\tau(1), \ldots, \tau(f(k_3) - 1)\} = \{\sigma(1), \ldots, \sigma(f(k_3) - 1)\} \subset \{1, \ldots, f(k_4) - 1\}.$$

Set $\tau(f(k_3))$ equal to the smallest $m \notin \{\sigma(1), \ldots, \sigma(f(k_3) - 1), \ \tau(f(k_2) + 1)\}$ equal to the next smallest, and so on. Then

$$\{\tau(1), \ldots, \tau(f(k_4) - 1)\} = \{1, \ldots, f(k_4) - 1\}$$

and

$$\sum_{n=1}^{f(k_4)-1} a_{\tau(n)} = \sum_{n=1}^{f(k_4)-1} a_n \leq s + \varepsilon.$$

Carrying on in this way, we construct a strictly increasing sequence $(k_i)_{i \geqslant 1}$ of positive integers and a permutation τ of $\mathbb{N}^+$, such that for each $j \geq 1$,

$$\sum_{n=1}^{f(k_{2j-1})-1} a_{\tau(n)} \geq t - \varepsilon \quad \text{and} \quad \sum_{n=1}^{f(k_{2j})-1} a_{\tau(n)} \leq s + \varepsilon.$$

If $i \in \mathbb{N}^+$ is even, then

$$\left| \sum_{n=f(k_i)}^{f(k_{i+1})-1} a_{\tau(n)} \right| \geqslant \sum_{n=1}^{f(k_{i+1})-1} a_{\tau(n)} - \sum_{n=1}^{f(k_i)-1} a_{\tau(n)}$$
$$\geq t - \varepsilon - (s + \varepsilon) = t - s - 2\varepsilon = \frac{1}{3}(t - s).$$

Similarly, if $i \in \mathbb{N}^+$ is odd, then

$$\left| \sum_{n=f(k_i)}^{f(k_{i+1})-1} a_{\tau(n)} \right| \geq \sum_{n=1}^{f(k_i)-1} a_{\tau(n)} - \sum_{n=1}^{f(k_{i+1})-1} a_{\tau(n)} > \frac{1}{3}(t-s).$$

Hence (1) holds. ■

Lemma 3 *Under the hypotheses of* Lemma 2, *the series* $\sum |a_n|$ *diverges.*

Proof. Construct the permutation τ and the sequence $(k_i)_{i \geqslant 1}$ in the proof of Lemma 2. Given $C > 0$, compute j such that $(j-1)|s-t| > 3C$. Then

$$\sum_{n=1}^{f(k_j)-1} \left| a_{\tau(n)} \right| = \sum_{i=1}^{j-1} \sum_{n=f(k_i)}^{f(k_{i+1})-1} \left| a_{\tau(n)} \right| \geqslant \sum_{i=1}^{j-1} \left| \sum_{n=f(k_i)}^{f(k_{i+1})-1} a_{\tau(n)} \right| \geq \frac{j-1}{3} |s-t| > C.$$

There exists M such that

$$\left\{ a_{\tau(1)}, \ldots, a_{\tau(f(k_j)-1)} \right\} \subset \{a_1, \ldots, a_M\}.$$

Then

$$\sum_{n=1}^{M} |a_n| \geqslant \sum_{n=1}^{f(k_j)-1} \left| a_{\tau(n)} \right| > C.$$

Since $C > 0$ is arbitrary, the conclusion follows. ■

Lemma 4 *Let* $\sum a_n$ *be a convergent series of real numbers, and* τ *a permutation of* $\mathbb{N}^+$ *such that* $\sum a_{\tau(n)}$ *diverges to infinity. Then it is impossible that* $\sum a_{\tau(n)}$ *have a convergent bracketing.*

Proof. Suppose there exists a bracketing $(f, \mathbf{b})$ of $\sum a_{\tau(n)}$ that converges to a sum s. Compute $N > 1$ such that

$$\sum_{n=1}^{\nu} a_{\tau(n)} > s + 1 \quad (\nu \geqslant N). \tag{2}$$

There exists $N_1 > N$ such that

$$\left| \sum_{n=1}^{f(N_1)-1} a_{\tau(n)} - s \right| = \left| \sum_{i=1}^{N_1-1} \sum_{n=f(i)}^{f(i+1)-1} a_{\tau(n)} - s \right| = \left| \sum_{i=1}^{N_1-1} b_i - s \right| < 1$$

and therefore

$$\sum_{n=1}^{f(N_1)-1} a_{\tau(n)} < s + 1.$$

Since $f(N_1) - 1 \geq N_1 - 1 \geq N$, this contradicts (2). ■

Lemma 5 *Let $\sum a_n$ be a weak-permutably convergent series of real numbers, and σ a permutation of $\mathbb{N}^+$. Then it is impossible that $\sum \left|a_{\sigma(n)}\right|$ diverge.*

Proof. Suppose that $\sum \left|a_{\sigma(n)}\right|$ does diverge. Then, by the full, extended version of **RST**$_2$, there is a permutation τ of $\mathbb{N}^+$ such that $\sum a_{\tau(n)}$ diverges to infinity. Since $\sum a_n$ is weak-permutably convergent, there exists a bracketing of $\sum a_{\tau(n)}$ that converges. This is impossible, in view of Lemma 4. ■

Arguing with classical logic, we see that if $\sum a_n$ is weak-permutably convergent, then, by Lemma 5, $\sum |a_n|$ must converge; whence $\sum a_n$ is permutably convergent, by **RST**$_1$.

Returning to intuitionistic logic, we have reached the **proof of Proposition 1**:

Proof. Suppose that there exists a bracketing of $\sum a_{\sigma(n)}$ that converges to a sum distinct from s. Then, by Lemma 3, $\sum |a_n|$ diverges. Lemma 5 shows that this is impossible. It follows from the tightness of the inequality on $\mathbb{R}$ that every convergent bracketing of $\sum a_{\sigma(n)}$ converges to s. ■

Since permutable convergence implies convergence and is a special case of weak-permutable convergence, we also have:

Corollary 6 *Let $\sum a_n$ be a permutably convergent series of real numbers, and let σ be a permutation of $\mathbb{N}^+$. Then $\sum a_{\sigma(n)} = \sum a_n$.*

3 BD-$\mathbb{N}$ and permutable convergence

A subset S of $\mathbb{N}^+$ is said to be **pseudobounded** if for each sequence[4] $(s_n)_{n\geqslant 1}$ in S there exists N such that $s_n/n < 1$ for all $n \geqslant N$—or, equivalently, if for each sequence $(s_n)_{n\geqslant 1}$ in S, $s_n/n \to 0$ as $n \to \infty$. Every bounded subset of $\mathbb{N}^+$ is pseudobounded; the converse holds classically, intuitionistically, and in recursive constructive mathematics, but Lietz et al. [15] and Lubarsky [16] have produced models of **BISH** in which it fails to hold for inhabited, countable, pseudobounded sets. Thus the principle

BD-N *Every inhabited, countable, pseudobounded subset of $\mathbb{N}^+$ is bounded*[5]

is independent of **BISH.** It is a serious problem of constructive reverse mathematics [10, Chapters 23–25] to determine which classical theorems are equivalent to BD-$\mathbb{N}$ over **BISH**. For example, it is known that the full form of the Open Mapping Theorem for Hilbert spaces is one such theorem; see [8, Theorem 5].

This section is devoted to our version of the **Riemann permutability theorem**:

Theorem 7 *In **BISH** + BD-$\mathbb{N}$, every permutably convergent series of real numbers is absolutely convergent.*

[4] In this definition, we can replace *sequence* by *increasing* (but not strictly increasing) *sequence*.

[5] **BD-N** was introduced by Ishihara in [12] (see also [17]).

Proof. Let $\sum_{i=1}^{\infty} a_i$ be a permutably convergent series of real numbers. Write

$$a_n^+ = \max\{a_n, 0\}, \ a_n^- = \max\{-a_n, 0\}.$$

To begin with, assume that $a_2^+ > 0$ and each a_i is rational. Given a positive rational number $\varepsilon < a_2^+$, define a binary mapping ϕ on $\mathbb{N}^+ \times \mathbb{N}^+$ such that

$$\phi(m,n) = 0 \Rightarrow m > n \wedge \sum_{i=n+1}^{m} a_i^+ \geqslant \varepsilon,$$

$$\phi(m,n) = 1 \Rightarrow m \leqslant n \vee \sum_{i=n+1}^{m} a_i^+ < \varepsilon.$$

Let

$$S \equiv \{n : \exists_m (\phi(m,n) = 0)\}.$$

Then $1 \in S$, and S is both countable and downward closed. In order to prove that S is pseudobounded, let $(s_n)_{n \geqslant 1}$ be an increasing sequence in S. We may assume that $s_1 = 1$. Define a map $\kappa : S \to \mathbb{N}^+$ by

$$\kappa(n) \equiv \min\left\{m : m > n \wedge \sum_{i=n+1}^{m} a_i^+ \geqslant \varepsilon\right\}.$$

Setting $\lambda_1 = 0$, we construct inductively a binary sequence $\lambda \equiv (\lambda_n)_{n \geqslant 1}$ and a mapping $\theta : \lambda^{-1}(1) \to \mathbb{N}^+$ such that for each $n \in \mathbb{N}^+$,

(a) if $\lambda_n = 0$ and $\lambda_{n+1} = 1$, then $n+1 \in S$;

(b) if $\lambda_n = 0 = \lambda_{n+1}$, then $s_{n+1} \leq n+1$;

(c) if $\lambda_n = 1$, then $\theta(n) = \min\{i \leqslant n : \forall_j (i \leqslant j \leqslant n \Rightarrow \lambda_j = 1)\}$;

(d) if $\lambda_n = 1$, then $\lambda_{n+1} = 0$ if and only if $n = \kappa(\theta(n))$.

Suppose we have defined $\lambda_1, \ldots, \lambda_n$ and, when $k \leq n$ and $\lambda_k = 1$, $\theta(k)$ with the applicable properties. In the case $\lambda_n = 0$, if $s_{n+1} \leqslant n+1$, we set $\lambda_{n+1} = 0$; and if $s_{n+1} > n+1$, we set $\lambda_{n+1} = 1$, noting that $n+1 \in S$ since S is downward closed. In the case $\lambda_n = 1$, since $\lambda_1 = 0$, we see that $\theta(n)$ is defined, that $\lambda_{\theta(n)-1} = 0$ and $\lambda_{\theta(n)} = 1$, and therefore, by (a), that $\theta(n) \in S$; whence $\kappa(\theta(n))$ is defined. We then set $\lambda_{n+1} = 0$ if $n = \kappa(\theta(n))$, and $\lambda_{n+1} = 1$ otherwise. This completes our inductive construction. Note that if $\lambda_n = 1$, then $\kappa(\theta(n)) \geq n$. For if $\theta(n) \leq \kappa(\theta(n)) < n$, then $\lambda_{\kappa(\theta(n))} = 1 = \lambda_{\kappa(\theta(n))+1}$; but by (d), $\lambda_{\kappa(\theta(n))+1} = 0$, a contradiction. Thus we have:

(e) if $\lambda_n = 1$, then $k = \kappa(\theta(n)) - n + 1 \geq 1$ and $\lambda_{n+k} = \lambda_{\kappa(\theta(n))+1} = 0$.

Note also that if $n \geq 2$, $\lambda_{n-1} = \lambda_{n+1} = 0$, and $\lambda_n = 1$, then by (a), $n \in S$, and by (d), $n = \kappa(\theta(n)) > n$, which is absurd.

For convenience, if $n \leq m$ and the following hold, we call the interval $I = [n, m]$ of $\mathbb{N}^+$ a *bad interval*:

— if $n > 1$ then $\lambda_{n-1} = 0$,

— $\lambda_{m+1} = 0$, and

— $\lambda_i = 1$ for all $i \in I$.

If $\lambda_{n-1} = 0$ and $\lambda_n = 1$, then $n \in S$, by (a), and $\theta(n) = n$; if also $\lambda_{n+1} = 0$, then $n = \kappa(\theta(n)) = \kappa(n) > n$, which is absurd. Thus there are no singleton bad intervals. We define a permutation σ of $\mathbb{N}^+$ as follows. If $\lambda_n = 0$ set $\sigma(n) \equiv n$. If $[n, m]$ is a bad interval (perforce with $m > n$), then by (c), $\theta(m) = n$, and (since $\lambda_{m+1} = 0$) using (d), we have $\kappa(\theta(m)) = \kappa(n) = m$; hence $\sum_{i=n+1}^{m} a_i^+ \geqslant \varepsilon$. Let σ map an initial segment $[n, n+k-1]$ of $[n, m]$ onto

$$\left\{i : n \leqslant i \leqslant m \wedge a_i^+ > 0\right\},$$

and map the remaining elements of $[n, m]$ onto

$$\left\{i : n \leqslant i \leqslant m \wedge a_i^+ = 0\right\}.$$

In this case,

$$\sum_{i=n+1}^{n+k-1} a_{\sigma(i)} = \sum_{i=n+1}^{n+k-1} a_i^+ = \sum_{i=n+1}^{m} a_i^+ \geq \varepsilon. \tag{3}$$

If $\lambda_n = 1$, then $\theta(n) \leq n$ and, by (e), $m \equiv \min\{k \geq 1 : \lambda_{n+k} = 0\}$ exists. It follows that $\theta(m) = \theta(n)$ and that $[\theta(n), m]$ is a bad interval containing n; whence σ is defined on $[\theta(n), m]$ and in particular at n. This completes the definition of σ, which is easily seen to be a permutation of $\mathbb{N}^+$.

Since $\sum_{i=1}^{\infty} a_{\sigma(i)}$ is convergent, there exists J such that $\sum_{i=j+1}^{k} a_{\sigma(i)} < \varepsilon$ whenever $J \leqslant j < k$. In view of (e), we can assume that $\lambda_J = 0$. If $n > J$ and $\lambda_n = 1$, then $\theta(n) > J$ and there exists $m > \theta(n)$ such that $[\theta(n), m]$ is a bad interval. Hence, by (3), there exists j with $J \leq \theta(n) \leq j \leq m$ such that $\sum_{i=\theta(n)+1}^{j} a_{\sigma(i)} \geq \varepsilon$, a contradiction. We conclude that $\lambda_{n-1} = 0 = \lambda_n$, and therefore $s_n \leq n$, for all $n \geq J+1$. Thus S is pseudobounded.

Applying BD-$\mathbb{N}$, we obtain a positive integer N such that $n < N$ for all $n \in S$. If $m > n \geqslant N$ and $\sum_{i=n+1}^{m} a_i^+ > \varepsilon$, then $\phi(m, n) \neq 1$, so $\phi(m, n) = 0$ and therefore $n \in S$, a contradiction. Hence $\sum_{i=n+1}^{m} a_i^+ \leqslant \varepsilon$ whenever $m > n \geqslant N$. Likewise, there exists N' such that $\sum_{i=n+1}^{m} a_i^- \leqslant \varepsilon$ whenever $m > n \geqslant N'$. Thus if $m > n \geqslant \max\{N, N'\}$, then

$$\sum_{i=n+1}^{m} |a_i| = \sum_{i=n+1}^{m} a_i^+ + \sum_{i=n+1}^{m} a_i^- \leqslant 2\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we conclude that the partial sums of the series $\sum |a_n|$ form a Cauchy sequence, and hence that the series converges.

It remains to remove the restrictions imposed in the second sentence of this proof. Pick $b_2 > 0$ such that $a_2 + b_2$ is positive and rational, and for each $i \neq 2$ pick b_i such that $0 < b_i < 2^{-i}$ and $a_i + b_i$ is rational. Note that the series $\sum_{i=1}^{\infty} b_i$

converges absolutely and so, by $\mathbf{RST}_1$, is permutably convergent. It reaily follows that $\sum_{i=1}^{\infty}(a_i + b_i)$ is permutably convergent and therefore, by the first part of this proof, absolutely convergent. Since $|a_i| \leq |a_i + b_i| + |b_i|$, the comparison test shows that $\sum_{i=1}^{\infty} |a_i|$ is convergent. ■

4 Weak-permutable convergence and BD-$\mathbb{N}$

Diener and Lubarsky [11] have constructed topological models showing that the absolute convergence of every permutably convergent series in $\mathbb{R}$ neither implies BD-$\mathbb{N}$ nor is provable within the Aczel-Rathjen CZF set-theoretic foundation [1] for **BISH**, and may therefore be of constructive reverse-mathematical significance in its own right. Their models lead us to ask: is there a variant of the Riemann permutability theorem that is *classically equivalent* to the original form and that implies BD-$\mathbb{N}$? Since weak-permutable and permutable convergence are classically equivalent, the main result of this section provides an affirmative answer:

Theorem 8 *The statement*

Every weak-permutably convergent series in $\mathbb{R}$ is absolutely convergent

implies BD-$\mathbb{N}$.

The hard part of the proof is isolated in the complicated construction in the following lemma.

Lemma 9 *Let $S \equiv \{s_1, s_2, \ldots\}$ be an inhabited, countable, pseudobounded subset of $\mathbb{N}$. Then there exists a sequence $(a_n)_{n\geqslant 1}$ of nonnegative rational numbers with the following properties.*

(i) $\sum (-1)^n a_n$ *is weak-permutably convergent.*

(ii) *If $\sum a_n$ converges, then S is bounded.*

Proof. First replace each s_n by $\max\{s_k : k \leqslant n\}$, thereby obtaining $s_1 \leqslant s_2 \leqslant \cdots$. Construct a binary sequence $(\lambda_k)_{k\geqslant 1}$ such that

$$\lambda_k = 0 \Rightarrow s_{2^{k+1}} = s_{2^k},$$
$$\lambda_k = 1 \Rightarrow s_{2^{k+1}} > s_{2^k}.$$

Setting $a_1 = 0$, let $a_n = \lambda_k / (n+1)$ whenever $k, n \in \mathbb{N}^+$ and $2^k \leqslant n < 2^{k+1}$. In order to show that $\sum_{n=1}^{\infty} (-1)^n a_n$ converges in $\mathbb{R}$, first observe that if $\lambda_k = 1$ and $2^k \leq m_1 \leqslant m_2 < 2^{k+1}$, then

$$\left| \sum_{n=m_1}^{m_2} (-1)^n a_n \right| = \left| \sum_{n=m_1}^{m_2} \frac{(-1)^n}{n+1} \right| \leq \frac{1}{m_1 + 1} < \frac{1}{2^k}.$$

On the other hand, if j, k, m_1, m_2 are positive integers with $2^k \le m_1 < 2^{k+1} \leqslant 2^j \le m_2 < 2^{j+1}$, then

$$\left|\sum_{n=m_1}^{m_2} (-1)^n a_n\right| \leqslant \left|\sum_{n=m_1}^{2^{k+1}-1} (-1)^n a_n\right| + \sum_{i=k+1}^{j-1} \left|\sum_{n=2^i}^{2^{i+1}-1} (-1)^n a_n\right| + \left|\sum_{n=2^j}^{m_2} (-1)^n a_n\right|$$

$$\le \frac{1}{2^k} + \sum_{i=k+1}^{j-1} \frac{1}{2^i} + \frac{1}{2^j} \le \sum_{i=k}^{\infty} \frac{1}{2^i} = \frac{1}{2^{k-1}}.$$

We now see that

$$\left|\sum_{n=m_1}^{m_2} (-1)^n a_n\right| \le \frac{1}{2^k} \qquad (m_2 \ge m_1 \ge 2^{k+1}). \tag{4}$$

It follows that the partial sums of $\sum_{n=1}^{\infty} (-1)^n a_n$ form a Cauchy sequence, and therefore the series converges to a sum $s \in \mathbb{R}$.

Consider any permutation σ of $\mathbb{N}^+$. In order to show that $\sum_{n=1}^{\infty} (-1)^{\sigma(n)} a_{\sigma(n)}$ has a convergent bracketing, we construct strictly increasing sequences $(j_k)_{k\geqslant 1}$ and $(n_k)_{k\geqslant 1}$ of positive integers such that for each k,

(a) $2^{j_k} < n_k < 2^{j_{k+1}}$,

(b) $\left\{1, 2, \ldots, 2^{j_k}\right\} \subset \{\sigma(n) : n \le n_k\} \subset \left\{1, 2, \ldots, 2^{j_{k+1}}\right\}$, and

(c) $\left|\sum_{n=j}^{i} (-1)^n a_n\right| < 2^{-k}$ for all $k \geqslant 1$ and $i \ge j \ge 2^{j_k}$.

First we set $j_1 = 2^2$ and choose $n_1 > 2^{j_1}$ such that

$$\{1, \ldots, 2^{j_1}\} \subset \{\sigma(n) : n \le n_1\}.$$

From (4) we have

$$\left|\sum_{n=j}^{i} (-1)^n a_n\right| < 2^{-1} \quad (i \ge j \ge 2^{j_1}).$$

Having found j_k and n_k such that $2^{j_k} < n_k$,

$$\left\{1, 2, \ldots, 2^{j_k}\right\} \subset \{\sigma(n) : n \le n_k\},$$

and

$$\left|\sum_{n=j}^{i} (-1)^n a_n\right| < 2^{-k} \qquad (i \ge j \ge 2^{j_k}),$$

choose $j_{k+1} > \max\{j_k, 2^{k+2}\}$ such that $n_k < 2^{j_{k+1}}$ and

$$\{\sigma(n) : n \le n_k\} \subset \left\{1, 2, \ldots, 2^{j_{k+1}}\right\}.$$

Then choose $n_{k+1} > 2^{j_{k+1}}$ such that

$$\left\{1, 2, \ldots, 2^{j_{k+1}}\right\} \subset \{\sigma(n) : n \leq n_{k+1}\}.$$

Since, by (4),

$$\left|\sum_{n=j}^{i} (-1)^n a_n\right| < 2^{-k-1} \quad (i \geq j \geq 2^{j_{k+1}}),$$

we have completed the inductive construction of the sequences $(j_k)_{k\geqslant 1}$ and $(n_k)_{k\geqslant 1}$ with properties (a)–(c).

Now consider the sequence $(s_{2^{j_k+1}})_{k\geqslant 1}$. Since S is pseudobounded, there exists a positive integer K_1 such that $s_{2^{j_k+1}} < k$ for all $k \geqslant K_1$. Suppose that for each positive integer $k \leqslant K_1$, there exists i_k such that $j_k \leqslant i_k < j_{k+1}$ and $\lambda_{i_k} = 1$. Then

$$s_{2^{i_1}} < s_{2^{i_2}} < \cdots < s_{2^{i_{K_1}}} < s_{2^{j_{K_1+1}}},$$

so $K_1 \leq s_{2^{j_{K_1+1}}}$; but $i_{K_1} > K_1$ and therefore $s_{2^{j_{K_1+1}}} < K_1$, a contradiction. Hence there exists $k_1 \leqslant K_1$ such that for each i with $j_{k_1} \leqslant i < j_{k_1+1}$, we have $\lambda_i = 0$, and therefore $a_n = 0$ whenever $2^i \leqslant n < 2^{i+1}$. Thus $a_n = 0$ whenever $2^{j_{k_1}} \leqslant n < 2^{j_{k_1+1}}$. It follows from this and (b) above that

$$\begin{aligned}
\left\{a_n : n \leq 2^{j_{k_1}}\right\} &\subset \left\{a_{\sigma(n)} : n \leq n_{k_1}\right\} \\
&\subset \left\{a_n : n \leq 2^{j_{k_1+1}}\right\} \\
&= \left\{a_n : n \leq 2^{j_{k_1}}\right\} \cup \left\{a_n : 2^{j_{k_1}} \leqslant n < 2^{j_{k_1+1}}\right\} \\
&= \left\{a_n : n \leq 2^{j_{k_1}}\right\} \cup \{0\} \\
&= \left\{a_n : n \leq 2^{j_{k_1}}\right\} \cup \{a_1\} \\
&= \left\{a_n : n \leq 2^{j_{k_1}}\right\}.
\end{aligned}$$

Hence

$$\left\{a_n : n \leq 2^{j_{k_1}}\right\} = \left\{a_{\sigma(n)} : n \leq n_{k_1}\right\}.$$

Next consider the sequence $\left(s_{2^{j_{k_1+k+1}}}\right)_{k\geqslant 1}$. Since S is pseudobounded, there exists a positive integer K_2 such that $s_{2^{j_{k_1+k+1}}} < k$ for all $k \geqslant K_2$. Suppose that for each positive integer $k \leqslant K_2$, there exists i_k such that $j_{k_1+k} \leqslant i_k < j_{k_1+k+1}$ and $\lambda_{i_k} = 1$. Then

$$s_{2^{i_1}} < s_{2^{i_2}} < \cdots < s_{2^{i_{K_2}}} < s_{2^{j_{k_1+K_2+1}}},$$

so $K_2 \leqslant s_{2^{j_{k_1+K_2+1}}} < K_2$, which is absurd. Hence there exists $\kappa \leqslant K_2$ such that for each i with $j_{k_1+\kappa} \leqslant i < j_{k_1+\kappa+1}$, we have $\lambda_i = 0$, and therefore $a_n = 0$ whenever $2^i \leqslant n < 2^{i+1}$. Setting $k_2 \equiv k_1 + \kappa$, we have $a_n = 0$ for all n with $2^{j_{k_2}} \leqslant n < 2^{j_{k_2+1}}$. It follows from this and (b) above that

$$\begin{aligned}\left\{a_n : n \leq 2^{j_{k_2}}\right\} &\subset \left\{a_{\sigma(n)} : n \leq n_{k_2}\right\} \\ &\subset \left\{a_n : n \leq 2^{j_{k_2}+1}\right\} \\ &= \left\{a_n : n \leq 2^{j_{k_2}}\right\} \cup \left\{a_n : 2^{j_{k_2}} \leqslant n+1 < 2^{j_{k_2+1}}\right\} \\ &= \left\{a_n : n \leq 2^{j_{k_2}}\right\} \cup \{0\} \\ &= \left\{a_n : n \leq 2^{j_{k_2}}\right\} \cup \{a_1\} \\ &= \left\{a_n : n \leq 2^{j_{k_2}}\right\}.\end{aligned}$$

Thus

$$\left\{a_n : n \leq 2^{j_{k_2}}\right\} = \left\{a_{\sigma(n)} : n \leq n_{k_2}\right\}.$$

Carrying on in this way, we construct positive integers $k_1 < k_2 < k_3 < \cdots$ such that for each i,

$$\left\{a_n : n \leq 2^{j_{k_i}}\right\} = \left\{a_{\sigma(n)} : n \leq n_{k_i}\right\}. \tag{5}$$

For each $i \in \mathbb{N}^+$ let

$$X_i \equiv \{n : 2^{j_{k_i}} < n \leq 2^{j_{k_{i+1}}},\ a_n \neq 0\}$$

and

$$Y_i \equiv \left\{\sigma(n) : n_{k_i} < n \leq n_{k_{i+1}},\ a_{\sigma(n)} \neq 0\right\}.$$

Observe that if $a_n = a_{n'} \neq 0$, then, choosing p, q such that $2^p \leq n < 2^{p+1}$ and $2^q \leq n' < 2^{q+1}$, we have $\lambda_p = \lambda_q = 1$, $a_n = 1/(n+1)$, and $a_{n'} = 1/(n'+1)$; hence $n = n'$. By (5), for each $n \in X_i$ there exists $m \leq n_{k_{i+1}}$ such that $a_n = a_{\sigma(m)}$; then $n = \sigma(m)$ and therefore $m = \sigma^{-1}(n)$. If $m \leq n_{k_i}$, then by (5), there exists $n' \leq 2^{j_{k_i}}$ such that $a_{n'} = a_{\sigma(m)} = a_n$ and therefore $n = n'$; but $n \in X_i$, so $n > 2^{j_{k_i}} \geq n'$, a contradiction. Hence $n_{k_i} < m$ and $\sigma(m) \in Y_i$. Similar arguments using (5) show that for each m, if $n_{k_i} < m \leq n_{k_{i+1}}$ and $a_{\sigma(m)} \neq 0$, then there exists $n \in X_i$ such that $a_{\sigma(m)} = a_n$ and therefore $n = \sigma(m)$. It readily follows that $n \rightsquigarrow \sigma(\sigma^{-1}(n))$ is a one-one mapping of X_i onto Y_i. Thus

$$\left|\sum_{m=n_{k_i}+1}^{n_{k_{i+1}}} (-1)^{\sigma(m)} a_{\sigma(m)}\right| = \left|\sum_{n=2^{j_{k_i}}+1}^{2^{j_{k_i+1}}} (-1)^n a_n\right| < \frac{1}{2^{k_i}},$$

the last inequality using (c) above. Hence

$$\sum_{i=1}^{\infty} \sum_{m=n_{k_i}+1}^{n_{k_{i+1}}} (-1)^{\sigma(m)} a_{\sigma(m)}$$

converges, by comparison with $\sum_{i=1}^{\infty} 2^{-k_i}$. Since σ is an arbitrary permutation of $\mathbb{N}^+$, it follows that $\sum_{n=1}^{\infty} a_n$ is weak-permutably convergent. This proves (i).

To prove (ii), suppose that $\sum_{n=1}^{\infty} a_n$ converges. There exists N_1 such that $\sum_{n=N_1}^{\infty} a_n < 1/4$. Also, there exists $N \geq N_1$ such that

$$\frac{2^n - 1}{2^{n+1} - 1} > \frac{1}{4} \quad (n \geq N).$$

If $n \geq N$ and $\lambda_n = 1$, then

$$\frac{1}{4} > \sum_{k=2^n}^{\infty} a_k \geq \sum_{k=2^n}^{2^{n+1}-1} a_k = \sum_{k=2^n}^{2^{n+1}-1} \frac{1}{k} > \frac{2^n - 1}{2^{n+1} - 1} > \frac{1}{4},$$

a contradiction. It follows that for all $n \geq N$ we have $\lambda_n = 0$ and therefore $s_n = s_{2^N}$. Hence $s_n \leqslant s_{2^N}$ for all n, and S is a bounded set. ■

The proof of Theorem 8 is now straightforward:

Proof. Given an inhabited, countable, pseudobounded subset S of $\mathbb{N}$, use Lemma 9 to construct a sequence $(a_n)_{n \geqslant 1}$ of nonnegative rational numbers such that $\sum (-1)^n a_n$ is weak-permutably convergent, and if $\sum (-1)^n a_n$ converges absolutely, then S is bounded. ■

5 Concluding remarks

We have shown that, over **BISH**,

— with BD-$\mathbb{N}$, every permutably convergent series is absolutely convergent;

— the absolute convergence of every weak-permutably convergent series implies BD-$\mathbb{N}$.

It follows from the latter result that if, in **BISH**, weak-permutable convergence implies, and is therefore equivalent to, permutable convergence, then the absolute convergence of every permutably convergent series implies, and is therefore equivalent to, BD-$\mathbb{N}$. Since the topological models in [11] show that this is not the case, we see that, within **BISH**, weak-permutable convergence is a strictly weaker notion than permutable convergence.

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References

[1] P. Aczel and M.J. Rathjen, *Constructive Set Theory,* in preparation; draft available at https://www1.maths.leeds.ac.uk/~rathjen/book.pdf

[2] R.A. Alps and D.S. Bridges, *Morse Set Theory—a Foundation for Constructive Mathematics*, in preparation, 2022–23.

[3] J. Berger and D.S. Bridges, 'Rearranging series constructively', J. Univ. Comp. Sci. **15**(17), 3160–3168, 2009.

[4] E. Bishop, *Foundations of Constructive Analysis*, McGraw-Hill, New York, 1967.

[5] E. Bishop and D.S. Bridges, *Constructive Analysis*, Grundlehren der Math. Wiss. **279**, Springer Verlag, Heidelberg, 1985.

[6] D.S. Bridges, 'A reverse look at Brouwer's fan theorem', in: *One Hundred Years of Intuitionism (1907–2007)* (Eds: van Atten, M.; Boldini, P.; Bourdeau, M.; Heinzmann, G.), Publications of the Henri Poincaré Archives, Birkhäuser, Basel, 316–325, 2008.

[7] D.S. Bridges and F. Richman, *Varieties of Constructive Mathematics*, London Math. Soc. Lecture Notes **97**, Cambridge Univ. Press, Cambridge, 1987.

[8] D.S. Bridges and H. Ishihara, 'A definitive constructive open mapping theorem?', Math. Logic Quarterly **44**(4), 545–552, 1998.

[9] D.S. Bridges and L.S. Vîţă, *Techniques of Constructive Analysis*, Universitext, Springer-Verlag, Heidelberg, 2006.

[10] D.S. Bridges, H. Ishihara, M.J. Rathjen, H. Schwichtenberg (eds), *Handbook of Constructive Mathematics*, Cambridge University Press, 2023.

[11] H. Diener and R. Lubarsky, 'Principles weaker than BD-$\mathbb{N}$', J. Symbolic Logic **78**(3), 873–885, 2013.

[12] H. Ishihara, 'Continuity properties in metric spaces', J. Symb. Logic **57**(2), 557–565, 1992.

[13] H. Ishihara and H. Diener, 'Bishop-Style Constructive Reverse Mathematics', in: *Handbook of Computability and Complexity in Analysis* (Vasco Brattka and Peter Hertling, eds), 347–365, Springer Cham, 2021.

[14] H. Ishihara, 'An Introduction to Constructive Reverse Mathematics', Chapter 23 of [10, pages 640–665], 2023.

[15] P. Lietz and T. Streicher, 'Realizability models refuting Ishihara's boundedness principle', Ann. Pure and Applied Logic **163**(12), 1803–1807, 2012.

[16] R. Lubarsky, 'On the failure of BD-$\mathbb{N}$ and BD, and an application to the anti-Specker property', J. Symbolic Logic **78**(1), 39–56, 2013.

[17] F. Richman, 'Intuitionistic notions of boundedness in $\mathbb{N}$', Math. Logic Quart. **55**(1), 31–36, 2009.

[18] G.F.B. Riemann, 'Ueber die Darstellbarkeit einer Function durch eine trigonometrische Reihe', in *Gesammelte Werke*, 227–264. Originally in: *Habilitationsschrift*, 1854, Abhandlungen der Königlichen Gesellschaft der Wissenschaften zu Göttingen, **13**.

[19] B. Spitters, 'Constructive results on operator algebras', J. Univ. Comp. Sci. **11**(12), 2096–2113, 2005.

[20] A.S. Troelstra, *Choice Sequences. A Chapter of Intuitionistic Mathematics*, Oxford Logic Guides, Clarendon Press, Oxford, 1977.

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