Density and choice for total continuous functionals*

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In his seminal paper *Interpretation of analysis by means of constructive functionals of finite types* in the volume 'Constructivity in Mathematics' edited by A. Heyting in 1959 Georg Kreisel has studied partial and total continuous higher order functionals. The paper states two important theorems on these notions:

- the density theorem, which says that any finite functional can be extended to a total one, and
- the choice principle for total continuous functionals, which says that whenever for any total x there is a total y such that Rxy, where the relation R is given by a total boolean–valued functional, then this dependency can be realized by a total functional f (which can be found uniformly in R).

Both of these theorems also have effective versions, in a natural sense of the word.

Kreisel's paper and the one by Kleene [Kle59] on *Countable functionals* in the same volume, which independently introduced equivalent concepts, have given rise to a lot of activity aimed at understanding and developing the mathematical notions involved. First of all there is the unpublished work of Tait in the so–called Stanford Report, which is based directly on Kreisel's work. Another early contribution of essential impact was Platek's thesis [Pla66], who had started his work with Kreisel and later also worked with Dana Scott. Particularly Scott, in a number of attempts [Sco70, Sco82], contributed a lot to the subject. His work influenced Yuri Ershov to come up with an elaborated theory, with a somewhat topological flavour [Ers74, Ers75, Ers77]. Many other people have also made essential contributions, among them Feferman, Gandy, Hyland, Normann and Berger, mostly again in a rather general context. In particular Normann's work [Nor93] can be viewed as establishing the density theorem in a more general setting.

My aim in this paper is to give complete proofs of the density theorem and the choice principle for total continuous functionals in the natural and concrete context of the partial continuous functionals [Ers77], essentially by specializing more general treatments in the literature. The proofs obtained are relatively short and hopefully perspicious, and may contribute to redirect attention to the fundamental questions Kreisel originally was interested in.

Obviously this work owes much to other sources. In particular I have made use of work by Scott [Sco82] (whose notion of an information system is taken as a basis to introduce domains), Roscoe [Ros87], Larsen and Winskel [LW84] and Berger [Ber93].

The paper is organized as follows. Section 1 treats information systems, and in section 2 it is shown that the partial orders defined by them are exactly the (Scott) domains with countable basis. Section 3 gives a characterization of the continuous functions between domains, in terms of approximable mappings. In section 4 cartesian products and function spaces of domains and information systems are introduced. In section 5 the partial and total continuous functionals are defined. Section 6 finally contains the proofs of the two theorems above; it will be clear that the same proofs also yield effective versions of these theorems.

1. Information systems

The basic idea of information systems is to provide an axiomatic setting for to describe approximations of abstract objects (like functions or functionals) by concrete, finite ones. We do not attempt to analyse the notion of 'concreteness' or finiteness here, but rather take an arbitrary countable set A of 'data objects' or 'tokens' as a basic notion to be explained axiomatically. In order to use such data objects to build approximations of abstract objects, we certainly need a notion of 'consistency', which determines when the elements of a finite set of data objects are consistent with each other. At our present level of generality there clearly is no way to define this notion concretely, hence it again must be described axiomatically. Finally we need an 'entailment relation' between consistent sets X of data objects and single data objects a, which expresses the fact that the information contained in X supersedes the information in a. The following axioms on these notions have been given by Scott in [Sco82].

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1.1 Definition. *An information system is defined to be a structure $(A, \mathrm{Con}, \vdash)$ where A is a countable set (the tokens), Con is a nonempty set of finite subsets of A (the consistent sets), and $\vdash$ is a subset of $\mathrm{Con} \times A$ (the entailment relation) which satisfy*

(i) $X \subseteq Y \in \mathrm{Con}$ *implies* $X \in \mathrm{Con}$;
(ii) $a \in A$ *implies* $\{a\} \in \mathrm{Con}$;
(iii) $X \vdash a$ *implies* $X \cup \{a\} \in \mathrm{Con}$;
(iv) $X \in \mathrm{Con}$ *and* $a \in X$ *implies* $X \vdash a$;
(v) $(X, Y \in \mathrm{Con}$ *and* $(\forall b \in Y) X \vdash b$ *and* $Y \vdash c)$ *implies* $X \vdash c$.

Any countable set A can be turned into an information system $\mathbf{A}$ by letting the set of data objects be A, $\mathrm{Con} = \{\emptyset\} \cup \{\{a\} : a \in A\}$ and $X \vdash a \iff a \in X$.

1.2 Definition. *The elements or ideals of an information system $\mathbf{A} = (A, \mathrm{Con}, \vdash)$ are defined to be those subsets z of A which satisfy*

(i) $X \subseteq^{\mathrm{fin}} z$ *implies* $X \in \mathrm{Con}$ *(z is consistent)*;
(ii) $X \subseteq^{\mathrm{fin}} z$ *and* $X \vdash a$ *implies* $a \in z$ *(z is deductively closed)*.

The set of all elements of $\mathbf{A}$ is written $|\mathbf{A}|$. It clearly follows that the intersection of any number of ideals is an ideal again.

1.3 Lemma. *Suppose $\mathbf{A} = (A, \mathrm{Con}, \vdash)$ is an information system. Then*

(i) $\emptyset \in \mathrm{Con}$;
(ii) $(X \in \mathrm{Con}$ *and* $Y \subseteq X$ *and* $Y \vdash a)$ *implies* $X \vdash a$.

Proof. (i) follows because Con is by assumption nonempty, and axiom (i). (ii) holds by axiom (v) since, by axiom (iv), $X \vdash b$ for each $b \in Y$. □

Suppose $\mathbf{A} = (A, \mathrm{Con}, \vdash)$ is an information system and $B \subseteq A$. We define $\overline{B}$, the *deductive closure* of B, by

$$\overline{B} = \{a \in A : X \vdash a \text{ for some } X \subseteq^{\mathrm{fin}} B\}.$$

$\overline{B}$ is the set of tokens deducible from B. Note that by part (ii) of the previous Lemma, if $X \in \mathrm{Con}$, then $\overline{X} = \{a \in A : X \vdash a\}$.

1.4 Lemma. *Suppose $\mathbf{A} = (A, \mathrm{Con}, \vdash)$ is an information system, $X \in \mathrm{Con}$ and Y a finite subset of A. Then*

(i) if $X \vdash a$ *for every* $a \in Y$, *then* $Y \in \mathrm{Con}$;
(ii) $\overline{X} \in |\mathbf{A}|$;
(iii) $\overline{\emptyset} \subseteq z$ *for every* $z \in |\mathbf{A}|$.

Proof. (i) We prove $X \cup Y \in \mathrm{Con}$ by induction on the size of Y; the result then follows by axiom (i). In case $Y = \emptyset$ there is nothing to prove, so suppose $Y = Y' \cup \{a\}$ and $X \cup Y' \in \mathrm{Con}$. Since $X \vdash a$ by assumption, we have $X \cup Y' \vdash a$ by Lemma 1.3(ii) and hence $X \cup Y \in \mathrm{Con}$ by axiom (iii).

(ii) If Y is a finite subset of $\overline{X}$, then $Y \in \mathrm{Con}$ by (i); hence $\overline{X}$ is consistent. If Y is a finite subset of $\overline{X}$ and $Y \vdash a$, then $X \vdash a$ by the definition of $\overline{X}$ and axiom (v); hence $\overline{X}$ is deductively closed.

(iii) Let $z \in |\mathbf{A}|$ and $a \in \overline{\emptyset}$, i.e. $\emptyset \vdash a$. Since z is deductively closed, it follows that $a \in z$. □

In the light of (i) it is convenient informally to allow '$\vdash$' to be used as a relation between Con and Con. So from now on '$X \vdash Y$' will mean '$X \vdash a$ for each $a \in Y$'.

1.5 Lemma. *If $\mathbf{A} = (A, \mathrm{Con}, \vdash)$ is an information system and $U \subseteq A$ is consistent, then $\overline{U} \in |\mathbf{A}|$.*

Proof. We first show that $\overline{U}$ is consistent. If $Y = \{a_1, a_2, \ldots, a_n\}$ is any finite subset of $\overline{U}$, then for each $i \in \{1, 2, \ldots, n\}$ there exists a finite subset Z_i of U such that $Z_i \vdash a$. But $Z = Z_1 \cup \cdots \cup Z_n$ is a finite subset of U, and so is in Con. Now $Z \vdash a_i$ for each i by Lemma 1.3(ii). Lemma 1.4(i) then tells us that $Y \in \mathrm{Con}$.

To show deductive closure, suppose that Y is a finite subset of $\overline{U}$ and that $Y \vdash b$. Exactly as above we can find a finite subset Z of U such that $Z \vdash Y$. But then $Z \vdash b$ by axiom (v), so $b \in \overline{U}$ as required. □

2. Complete partial orders and domains

Let $(I, \leq)$ be a partial order, i.e. $\leq$ is a reflexive and transitive subset of $I \times I$ satisfying $i \leq j \leq i \implies i = j$ (antisymmetry). We say that I is *directed* if it is nonempty and for any $i, j \in I$ there is a $k \in I$ such that $i \leq k$ and $j \leq k$. A partial order $(D, \leq)$ having a least element $\bot$ is said to be *complete* (and we say that D is a *complete partial order*, abbreviated to *cpo*) if every directed subset $M \subseteq D$ has a least upper bound $\bigsqcup M$. A point x of a cpo D is said to be *compact* or *finite* if, for every directed collection $M \subseteq D$ such that $x \leq \bigsqcup M$, there is a $y \in M$ such that $x \leq y$. Let $\mathbf{B}_D$ denote the collection of compact elements of D; $\mathbf{B}_D$ is called the *basis* of D. The cpo D is *algebraic* if, for every $x \in D$, the set $M = \{x_0 \in \mathbf{B}_D : x_0 \leq x\}$ is directed and $x = \bigsqcup M$. A cpo D is *bounded complete* or *consistently complete* if every bounded subset of D (or equivalently every bounded finite subset of D) has a least upper bound. We call bounded complete algebraic cpo's *Scott domains* or just *domains*.

With any information system $\mathbf{A} = (A, \mathrm{Con}, \vdash)$ we can associate the partial order $(|\mathbf{A}|, \subseteq)$. Our aim in this section is to show that the partial orders obtained in this way are exactly the Scott domains with countable basis.

2.1 Theorem. *Suppose* $\mathbf{A} = (A, \mathrm{Con}, \vdash)$ *is an information system. Then* $(|\mathbf{A}|, \subseteq)$ *is a domain with the countable basis* $\{\overline{X} : X \in \mathrm{Con}\}$.

Proof. $(|\mathbf{A}|, \subseteq)$ clearly is a partial order. It has $\overline{\emptyset}$ as its least element by Lemma 1.4(iii). To prove completeness, let $M \subseteq |\mathbf{A}|$ be directed. Then $\bigcup M$ is consistent and deductively closed and hence is the least upper bound of M.

We now show that, for any $z \in |\mathbf{A}|$, that z is compact if and only if $z = \overline{X}$ for some $X \in \mathrm{Con}$. First assume that z is compact. Clearly $M := \{\overline{X} : X \subseteq^{\mathrm{fin}} z\}$ is directed and $z = \bigcup M$. Since by assumption z is compact, there is a finite subset X of z such that $z \subseteq \overline{X}$, hence $z = \overline{X}$. Conversely, let $X = \{a_1, a_2, \ldots, a_n\} \in \mathrm{Con}$ and $M \subseteq |\mathbf{A}|$ directed such that $\overline{X} \subseteq \bigcup M$. Then $a_i \in z_i$ for some $z_i \in M$. Since M is directed, there is $z^* \in M$ such that $z_i \subseteq z^*$, hence $X \subseteq z^*$ and therefore $\overline{X} \subseteq z^*$. This means that $\overline{X}$ is compact.

The algebraicity of $(|\mathbf{A}|, \subseteq)$ now is obvious since, for every $z \in |\mathbf{A}|$, $\{\overline{X} : X \in \mathrm{Con} \text{ and } \overline{X} \subseteq z\}$ clearly is directed and has z as its least upper bound.

Finally every bounded subset $M \subseteq |\mathbf{A}|$ has $\overline{\bigcup M}$ as its least upper bound, and hence $(|\mathbf{A}|, \subseteq)$ is bounded complete. $\square$

We now prove that any domain with countable basis can in fact be represented in this way, as the domain of an appropriate information system associated with it. To obtain this result we need a Lemma saying that in a cpo least upper bounds of finite sets of compact elements are compact, when they exist.

2.2 Lemma. *Suppose* $(D, \leq)$ *is a cpo,* $X \subseteq^{\mathrm{fin}} \mathbf{B}_D$ *and* $\bigsqcup X$ *exists. Then* $\bigsqcup X \in \mathbf{B}_D$.

Proof. Suppose $M \subseteq D$ is directed and $\bigsqcup X \leq \bigsqcup M$. Let $X = \{e_1, e_2, \ldots, e_n\}$. Since $e_i \leq \bigsqcup M$ and $e_i \in \mathbf{B}_D$ there is a $x_i \in M$ with $e_i \leq x_i$. Since M is directed, there is a $x^* \in M$ such that $x_i \leq x^*$ for every $i \in \{1, 2, \ldots, n\}$. Hence $\bigsqcup X \leq x^*$. $\square$

2.3 Definition. *Let* D *be a domain with countable basis* $\mathbf{B}_D$. *Define* $ID = (\mathbf{B}_D, \mathrm{Con}, \vdash)$, *where* Con *and* $\vdash$ *are defined by*

$$X \in \mathrm{Con} \iff X \subseteq^{\mathrm{fin}} \mathbf{B}_D \text{ and } X \text{ is bounded in } D;$$
$$X \vdash e \iff X \in \mathrm{Con} \text{ and } e \leq \bigsqcup X.$$

Let D and E be partial orders. A function $f\colon D \to E$ is *monotone* if $d \leq d'$ implies $f(d) \leq f(d')$, and *orderpreserving* if $d \leq d' \iff f(d) \leq f(d')$. A bijective orderpreserving function $f\colon D \to E$ is called an *isomorphism*. Now let D and E be cpo's. f is *continuous* if it is monotone and if for any directed set $M \subseteq D$

$$f(\bigsqcup M) = \bigsqcup\{f(d) : d \in M\}.$$

Note that from the monotonicity of f the directedness of $\{f(d) : d \in M\}$ follows, and hence the supremum on the right hand side exists in E. Note also that any surjective orderpreserving function $f\colon D \to E$ must be injective, hence an isomorphism and therefore also continuous.

It is easy to see that the concatenation of two continuous functions is continuous again.

2.4 Lemma. *Suppose D, E and F are cpo's. Let $f: D \to E$ and $g: E \to F$ be continuous functions. Then $g \circ f: D \to F$ is continuous.*

Proof. Clearly $g \circ f$ is monotone. Furthermore we have that $\bigsqcup_{d \in M} g(f(d)) = g(\bigsqcup_{d \in M} f(d)) = g(f(\bigsqcup M))$. $\square$

2.5 Theorem. *Let D be a domain with countable basis. Then ID is an information system with domain of elements $|ID|$ isomorphic to D. The isomorphism pair is*

$$\varphi: D \to |ID| \quad \textit{given by} \quad \varphi(d) = \{e \in \mathbf{B}_D : e \leq d\},$$

$$\psi: |ID| \to D \quad \textit{given by} \quad \psi(z) = \bigsqcup z.$$

Proof. It is easy to see that ID is an information system.

φ is well–defined: We have to show that $\varphi(d) \in |ID|$. Let us first prove that $\varphi(d)$ is consistent. So let $e_1, e_2, \dots, e_n \leq d$. We have to show that $\{e_1, e_2, \dots, e_n\} \in \mathrm{Con}$. But this is clear, since $\{e_1, e_2, \dots, e_n\}$ is bounded by d. We now prove that $\varphi(d)$ is deductively closed. Choose again $e_1, e_2, \dots, e_n \leq d$ and let $e \leq \bigsqcup\{e_1, e_2, \dots, e_n\}$. We have to show $e \in \varphi(d)$, i.e. $e \leq d$. But this follows from $e_1, e_2, \dots, e_n \leq d$.

ψ is well–defined: Let $z \in |ID|$. By Theorem 2.1 it suffices to show that z is directed. So let $X \subseteq^{\mathrm{fin}} z$. Then $X \subseteq \mathbf{B}_D$ and $X \in \mathrm{Con}$, hence X is bounded in D, hence $\bigsqcup X$ exists. Now by Lemma 2.2 $\bigsqcup X \in \mathbf{B}_D$. Therefore $\bigsqcup X \in z$ by the deductive closure of z.

$\psi \circ \varphi = \mathrm{id}_D$: Since D is algebraic, we have that $\varphi(d) = \{e \in \mathbf{B}_D : e \leq d\}$ is directed and $d = \bigsqcup \varphi(d)$.

$\varphi \circ \psi = \mathrm{id}_{|ID|}$: We have to show that $\varphi(\bigsqcup z) = z$ for every $z \in |ID|$. To prove $\supseteq$, let $z \subseteq \mathbf{B}_D$ be consistent and deductively closed. Recall that $\varphi(\bigsqcup z) = \{e \in \mathbf{B}_D : e \leq \bigsqcup z\}$. Now let $e \in z$. Then $e \leq \bigsqcup z$, hence $e \in \varphi(\bigsqcup z)$. To prove $\subseteq$, let $z \in |ID|$ and $e \in \varphi(\bigsqcup z)$, i.e. $e \leq \bigsqcup z$. We must show $e \in z$. Now since z is directed, from $e \leq \bigsqcup z$ and $e \in \mathbf{B}_D$ we can conclude $e \leq e'$ for some $e' \in z$. Since z is deductively closed and $e \leq \bigsqcup\{e'\}$, it follows that $e \in z$.

φ, ψ clearly are monotone. Hence φ and ψ are continuous and therefore form an isomorphism pair. $\square$

3. Mappings on information systems

Let $\mathbf{A} = (A, \mathrm{Con}_A, \vdash_A)$ and $\mathbf{B} = (B, \mathrm{Con}_B, \vdash_B)$ be information systems. We want to study 'information respecting' mappings from $\mathbf{A}$ into $\mathbf{B}$. Such a mapping is given by a relation r between Con_A and B, where Xrb intuitively means that whenever we are given the information $X \in \mathrm{Con}_A$ on the argument, then we know that at least the data object b belongs to the value.

3.1 Definition. *Let $\mathbf{A} = (A, \mathrm{Con}_A, \vdash_A)$ and $\mathbf{B} = (B, \mathrm{Con}_B, \vdash_B)$ be information systems. A relation $r \subseteq \mathrm{Con}_A \times B$ is an approximable mapping if it satisfies*
(i) Xrb_i for all $i \in \{1, 2, \dots, n\}$ implies $\{b_1, b_2, \dots, b_n\} \in \mathrm{Con}_B$;
(ii) Xrb_i for all $i \in \{1, 2, \dots, n\}$ and $\{b_1, b_2, \dots, b_n\} \vdash_B b$ implies Xrb;
(iii) $X \vdash_A X'$ and $X'rb$ implies Xrb.

We write $r: \mathbf{A} \to \mathbf{B}$ to mean that r is an approximable mapping from $\mathbf{A}$ to $\mathbf{B}$.

3.2 Theorem. *Let $\mathbf{A} = (A, \mathrm{Con}_A, \vdash_A)$ and $\mathbf{B} = (B, \mathrm{Con}_B, \vdash_B)$ be information systems. With any approximable mapping $s: \mathbf{A} \to \mathbf{B}$ we can associate a continuous function $|s|: |\mathbf{A}| \to |\mathbf{B}|$ by*

$$|s|(z) := \{b \in B : Xsb \textit{ for some } X \subseteq^{\mathrm{fin}} z\}.$$

Conversely, with any continuous function $f: |\mathbf{A}| \to |\mathbf{B}|$ we can associate an approximable mapping $r_f: \mathbf{A} \to \mathbf{B}$ by

$$X r_f b :\iff b \in f(\overline{X}).$$

These assignments are inverse to each other, i.e. $f = |r_f|$ and $s = r_{|s|}$.

Proof. We first show that $|s|$ is well–defined. So let $z \in |\mathbf{A}|$.

$|s|(z)$ is consistent. Let $b_1, b_2, \dots, b_n \in |s|(z)$. Then there are $X_1, X_2, \dots, X_n \subseteq^{\text{fin}} z$ such that $X_i s b_i$. Hence $X := X_1 \cup X_2 \cup \dots \cup X_n \subseteq^{\text{fin}} z$ and $X s b_i$ by Definition 3.1(iii). Now from Definition 3.1(i) we can conclude that $\{b_1, b_2, \dots, b_n\} \in \mathrm{Con}_B$.

$|s|(z)$ is deductively closed. Let $b_1, b_2, \dots, b_n \in |s|(z)$ and $\{b_1, b_2, \dots, b_n\} \vdash_B b$. We must show $b \in |s|(z)$. As before we find $X \subseteq^{\text{fin}} z$ such that $X s b_i$. Now from Definition 3.1(ii) we can conclude $X s b$ and hence $b \in |s|(z)$.

The monotonicity of $|s|$ clearly follows from the definition. To see that $|s|\colon |\mathbf{A}| \to |\mathbf{B}|$ is continuous, let $M \subseteq |\mathbf{A}|$ be directed. Then we have

$$\begin{aligned} |s|(\bigcup M) &= \{b \in B : (\exists X \subseteq^{\text{fin}} \bigcup M) X s b\} \\ &= \{b \in B : (\exists z \in M)(\exists X \subseteq^{\text{fin}} z) X s b\} && \text{since } M \text{ is directed} \\ &= \bigcup_{z \in M} \{b \in B : (\exists X \subseteq^{\text{fin}} z) X s b\} \\ &= \bigcup_{z \in M} |s|(z). \end{aligned}$$

Now let $f\colon |\mathbf{A}| \to |\mathbf{B}|$ be continuous. It is easy to verify that r_f is indeed an approximable mapping. Furthermore

$$\begin{aligned} f(z) &= f(\bigcup_{X \subseteq^{\text{fin}} z} \overline{X}) && \text{since } |\mathbf{A}| \text{ is algebraic} \\ &= \bigcup_{X \subseteq^{\text{fin}} z} f(\overline{X}) && \text{since f is continuous} \\ &= \{b \in B : (\exists X \subseteq^{\text{fin}} z) b \in f(\overline{X})\} \\ &= \{b \in B : (\exists X \subseteq^{\text{fin}} z) X r_f b\} \\ &= |r_f|(z). \end{aligned}$$

Finally, for any approximable mapping $s\colon \mathbf{A} \to \mathbf{B}$ we have

$$\begin{aligned} X s b &\iff (\exists Y \subseteq^{\text{fin}} \overline{X}) Y s b && \text{by 3.1(iii)} \\ &\iff b \in |s|(\overline{X}) \\ &\iff X r_{|s|} b. \end{aligned}$$

□

4. Products and function spaces

We discuss some canonical ways to construct new cpo's from given ones:

- the cartesian product $D \times E$, consisting of all pairs (d, e) with $d \in D$ and $e \in E$ under the component–wise order, and
- the function space $D \to E$, consisting of all continuous functions from D to E under the pointwise order.

We introduce these two constructs, and define operations on information systems corresponding to them.

The *cartesian product* of two cpo's $(D, \leq_D)$ and $(E, \leq_E)$ is defined to be $(D \times E, \leq)$ with the *component–wise order* $\leq$, i.e. $(d, e) \leq (d', e') \iff d \leq_D d'$ and $e \leq_E e'$. We want to show that $(D \times E, \leq)$ is a cpo again. For the proof we need to know that the supremum of a directed subset $M \subseteq D \times E$ operates component–wise. Denoting the left and right projection functions by π_{left} and π_{right} we have

4.1 Lemma. *Suppose D and E are cpo's. Let $M \subseteq D \times E$ be directed. Then $\pi_{\text{left}}(M)$ and $\pi_{\text{right}}(M)$ are directed too, and $\bigsqcup M = (\bigsqcup \pi_{\text{left}}(M), \bigsqcup \pi_{\text{right}}(M))$.*

Proof. We first show that e.g. $\pi_{\text{left}}(M)$ is directed. So let $d, d' \in \pi_{\text{left}}(M)$. Then there are $e, e' \in E$ such that $(d, e), (d', e') \in M$. Since M is directed, we have $(d, e), (d', e') \leq (d'', e'')$ for some $(d'', e'') \in M$. Therefore $d, d' \leq_D d'' \in \pi_{\text{left}}(M)$.

Hence $(\bigsqcup \pi_{\text{left}}(M), \bigsqcup \pi_{\text{right}}(M))$ exists. Clearly this pair is an upper bound of M. It is also a least upper bound. To see this suppose that (d', e') is another upper bound, i.e. $(d, e) \leq (d', e')$ for all $(d, e) \in M$. But then $d \leq_D d'$ for all $d \in \pi_{\text{left}}(M)$, i.e. $\bigsqcup \pi_{\text{left}}(M) \leq_D d'$. □

4.2 Corollary. *Suppose D and E are cpo's. Then the set $D \times E$ equipped with the component–wise order is a cpo again.* $\square$

Note that the continuity of π_{left} and π_{right} also follows from Lemma 4.1.

We now define a corresponding operation on information systems, assigning to any $\mathbf{A} = (A, \mathrm{Con}_A, \vdash_A)$ and $\mathbf{B} = (B, \mathrm{Con}_B, \vdash_B)$ an $\mathbf{A} \times \mathbf{B} = (C, \mathrm{Con}, \vdash)$ such that $|\mathbf{A} \times \mathbf{B}| \cong |\mathbf{A}| \times |\mathbf{B}|$. Without loss of generality we may assume that A and B are disjoint. But then any pair (x, y) of elements $x \in |\mathbf{A}|$ and $y \in |\mathbf{B}|$ can be approximated in each component separately. Hence we choose as data objects in $\mathbf{A} \times \mathbf{B}$ simply the union $C := A \cup B$. Consistency and entailment then is inherited in the obvious way from $\mathbf{A}$ and $\mathbf{B}$: $X \in \mathrm{Con} \iff X \cap A \in \mathrm{Con}_A$ and $X \cap B \in \mathrm{Con}_A$, and $X \vdash c \iff (c \in A \implies X \cap A \vdash_A c)$ and $(c \in B \implies X \cap B \vdash_B c)$. The elements of $\mathbf{A} \times \mathbf{B}$ then are exactly the unions of the elements of $\mathbf{A}$ and of $\mathbf{B}$: $|\mathbf{A} \times \mathbf{B}| = \{x \cup y : x \in |\mathbf{A}| \text{ and } y \in |\mathbf{B}|\}$.

4.3 Lemma. *If $\mathbf{A} = (A, \mathrm{Con}_A, \vdash_A)$ and $\mathbf{B} = (B, \mathrm{Con}_B, \vdash_B)$ are information systems with $A \cap B = \emptyset$, then $\mathbf{A} \times \mathbf{B}$ defined as above is an information system, and*

$$|\mathbf{A} \times \mathbf{B}| = \{x \cup y : x \in |\mathbf{A}| \text{ and } y \in |\mathbf{B}|\} \cong |\mathbf{A}| \times |\mathbf{B}|. \qquad \square$$

4.4 Corollary. *Suppose D and E are domains with countable bases. Then the set $D \times E$ equipped with the component–wise order is again a domain with countable basis.*

Proof. $D \times E \cong |ID| \times |IE| \cong |ID \times IE|$. $\square$

The *function space* of two cpo's $(D, \leq_D)$ and $(E, \leq_E)$ is defined to be $(D \to E, \leq)$, where $D \to E$ is the set of all continuous functions from D to E, and the order $\leq$ is defined pointwise, i.e. $f \leq g \iff \forall d\; f(d) \leq g(d)$. We want to show that $(D \to E, \leq)$ is a cpo again. This will follow from the observation that the supremum of a directed subset of $D \to E$ is formed pointwise. To prove this we will make use of the fact that the order of forming least upper bounds is irrelevant provided all relevant suprema exist. More precisely we have

4.5 Lemma. *Suppose D is a cpo and $d_{ij} \in D$ for all $i \in I$ and $j \in J$. Assume that*

(i) $\bigsqcup_{i \in I} d_{ij}$ exists for any $j \in J$,
(ii) $\bigsqcup_{j \in J} d_{ij}$ exists for any $i \in I$, and
(iii) $\bigsqcup_{i \in I} \bigsqcup_{j \in J} d_{ij}$ exists.

Then also $\bigsqcup_{j \in J} \bigsqcup_{i \in I} d_{ij}$ exists, and we have $\bigsqcup_{i \in I} \bigsqcup_{j \in J} d_{ij} = \bigsqcup_{j \in J} \bigsqcup_{i \in I} d_{ij}$.

Proof. We first prove that $\bigsqcup_{i \in I} \bigsqcup_{j \in J} d_{ij}$ is an upper bound of all $\bigsqcup_{i \in I} d_{ij}$. But this is clear, since $d_{ij} \leq \bigsqcup_{j \in J} d_{ij}$ for all $i \in I$ and hence $\bigsqcup_{i \in I} d_{ij} \leq \bigsqcup_{i \in I} \bigsqcup_{j \in J} d_{ij}$.

It remains to prove that $\bigsqcup_{i \in I} \bigsqcup_{j \in J} d_{ij}$ is the least upper bound of all $\bigsqcup_{i \in I} d_{ij}$. So assume that d is some upper bound. We must show $\bigsqcup_{i \in I} \bigsqcup_{j \in J} d_{ij} \leq d$. But this is clear, since $d_{ij} \leq \bigsqcup_{i \in I} d_{ij} \leq d$ for any $i \in I$ and $j \in J$ by assumption. $\square$

4.6 Lemma. *Suppose D and E are cpo's. Let $\{f_i : i \in I\} \subseteq D \to E$ be directed. Then $f(d) := \bigsqcup_{i \in I} f_i(d)$ is a well–defined continuous function from D to E, and f is the least upper bound of $\{f_i : i \in I\}$.*

Proof. Suppose $d \in D$. Then $\{f_i(d) : i \in I\}$ is directed since $\{f_i : i \in I\}$ is, hence $\bigsqcup_{i \in I} f_i(d)$ exists, i.e. f is well–defined. f is continuous, for if $\{d_j : j \in J\}$ is directed, we have

$$f(\bigsqcup_{j \in J} d_j) = \bigsqcup_{i \in I} \bigsqcup_{j \in J} f_i(d_j) = \bigsqcup_{j \in J} \bigsqcup_{i \in I} f_i(d_j) = \bigsqcup_{j \in J} f(d_j),$$

where the middle equality follows from Lemma 4.5.

It remains to be shown that f is the least upper bound of $\{f_i : i \in I\}$. Clearly f is an upper bound, since $f_i(d) \leq \bigsqcup_{i \in I} f_i(d) = f(d)$ for any $d \in D$. It is also the least upper bound, for if g is any other upper bound, we have $f_i(d) \leq g(d)$ and hence $f(d) = \bigsqcup_{i \in I} f_i(d) \leq g(d)$ for any $d \in D$. $\square$

4.7 Corollary. *$D \to E$ is a cpo if D and E are.* □

We now define a corresponding operation on information systems, assigning to any $\mathbf{A} = (A, \mathrm{Con}_A, \vdash_A)$ and $\mathbf{B} = (B, \mathrm{Con}_B, \vdash_B)$ an $\mathbf{A} \to \mathbf{B} = (C, \mathrm{Con}, \vdash)$ such that $|\mathbf{A} \to \mathbf{B}| \cong |\mathbf{A}| \to |\mathbf{B}|$. We first have to decide on the set of data objects. The simplest finite piece of information on a continuous function from $|\mathbf{A}|$ to $|\mathbf{B}|$ is to say that if we are given the information $X \in \mathrm{Con}_A$ on the argument, then this suffices to know at least the information $b \in B$ on the value. Hence we let $C := \mathrm{Con}_A \times B$. Now when a set $\{(X_1, b_1), \ldots, (X_n, b_n)\}$ of such data objects is to be called consistent? Clearly just in the case when for any subset $I \subseteq \{1, 2, \ldots, n\}$ such that the left hand sides $\bigcup\{X_i : i \in I\}$ are consistent, i.e. may be viewed as approximations to a common argument, then the corresponding right hand sides $\{b_i : i \in I\}$ should be consistent too, i.e. it should be possible to view them as approximations to a single value. Therefore Con is defined to be the set of all $\{(X_1, b_1), \ldots, (X_n, b_n)\} \subseteq C$ such that

$$\forall I \subseteq \{1, \ldots, n\}. \bigcup_{i \in I} X_i \in \mathrm{Con}_A \implies \{b_i : i \in I\} \in \mathrm{Con}_B.$$

For the definition of the entailment relation $\vdash$ it is helpful to first define the notion of an *application* of $W := \{(X_1, b_1), \ldots, (X_n, b_n)\} \in \mathrm{Con}$ to $X \in \mathrm{Con}_A$:

$$\{(X_1, b_1), \ldots, (X_n, b_n)\}X := \{b_i : X \vdash_A X_i\}.$$

From the definition of Con we know that this set is in Con_B. Clearly application is *monotone in the second argument*, in the sense that

$$X \vdash_A X' \implies WX \vdash_B WX'.$$

In fact we even have $WX' \subseteq WX$ if $X \vdash_A X'$. Now define $W \vdash (X, b)$ by

$$W \vdash (X, b) :\iff WX \vdash_B b.$$

4.8 Lemma. *If $\mathbf{A}$ and $\mathbf{B}$ are information systems, then so is $\mathbf{A} \to \mathbf{B}$ defined as above.*

Proof. Let $\mathbf{A} = (A, \mathrm{Con}_A, \vdash_A)$ and $\mathbf{B} = (B, \mathrm{Con}_B, \vdash_B)$. The axioms 1.1(i), (ii) and (iv) are clearly satisfied. For (iii), suppose $\{(X_1, b_1), \ldots, (X_n, b_n)\} \vdash (X, b)$, i.e. $\{b_j : X \vdash_A X_j\} \vdash_B b$. We have to show that $\{(X_1, b_1), \ldots, (X_n, b_n), (X, b)\} \in \mathrm{Con}$. So let $I \subseteq \{1, \ldots, n\}$ and suppose $X \cup \bigcup_{i \in I} X_i \in \mathrm{Con}_A$. We must show that $b \cup \{b_i : i \in I\} \in \mathrm{Con}_B$. Let $J \subseteq \{1, \ldots, n\}$ consist of those j with $X \vdash_A X_j$. Then also $X \cup \bigcup_{i \in I} X_i \cup \bigcup_{j \in J} X_j \in \mathrm{Con}_A$. Since $\bigcup_{i \in I} X_i \cup \bigcup_{j \in J} X_j \in \mathrm{Con}_A$, from the consistency of $\{(X_1, b_1), \ldots, (X_n, b_n)\}$ we can conclude that $\{b_i : i \in I\} \cup \{b_j : j \in J\} \in \mathrm{Con}_B$. But $\{b_j : j \in J\} \vdash_B b$ by assumption. Hence $\{b_i : i \in I\} \cup \{b_j : j \in J\} \cup \{b\} \in \mathrm{Con}_B$.

For (v), suppose $W \vdash \{(X_1, b_1), \ldots, (X_n, b_n)\} \vdash (X, b)$. We have to show that $WX \vdash_B b$. Note that from $X \vdash_A X_i$ we can conclude $WX \vdash_B WX_i$ by the monotonicity of application in the second argument. Hence $WX \vdash_B \bigcup\{WX_i : X \vdash_A X_i\} \vdash_B \{b_i : X \vdash_A X_i\} \vdash_B b$ □

Note that with the above definitions of the entailment relation $\vdash$ in $\mathbf{A} \to \mathbf{B}$ application is also *monotone in the first argument*, i.e.

$$W \vdash W' \implies WX \vdash_B W'X.$$

To see this let $W' = \{(X_1, b_1), \ldots, (X_n, b_n)\}$ and observe that $WX \vdash_B \bigcup\{WX_i : X \vdash_A X_i\} \vdash_B \{b_i : X \vdash_A X_i\} = W'X$.

4.9 Lemma. *Let $\mathbf{A}$ and $\mathbf{B}$ be information systems. Then the elements of $\mathbf{A} \to \mathbf{B}$ are exactly the approximable mappings from $\mathbf{A}$ to $\mathbf{B}$.*

Proof. Let $\mathbf{A} = (A, \mathrm{Con}_A, \vdash_A)$ and $\mathbf{B} = (B, \mathrm{Con}_B, \vdash_B)$. Suppose $r \in |\mathbf{A} \to \mathbf{B}|$. Then $r \subseteq \mathrm{Con}_A \times B$ is consistent and deductively closed. We have to show that r satisfies the axioms 3.1(i)–(iii) of approximable mappings.

(i). Let Xrb_i for $i \in \{1, \ldots, n\}$. We must show that $\{b_1, \ldots, b_n\} \in \mathrm{Con}_B$. But this clearly follows from the consistency of r.

(ii). Let Xrb_i for $i \in \{1, \dots, n\}$ and $\{b_1, \dots, b_n\} \vdash_B b$. We must show that Xrb. But by the definition of $\vdash$ in $\mathbf{A} \to \mathbf{B}$ we have $\{(X, b_1), \dots, (X, b_n)\} \vdash (X, b)$, hence Xrb by the deductive closure of r.

(iii). Let $X \vdash_A X'$ and $X'rb$. We must show that Xrb. But $\{(X', b)\} \vdash (X, b)$ since $\{(X', b)\}X = \{b\}$ (which follows from $X \vdash_A X'$), hence again Xrb by the deductive closure of r.

For the other direction suppose that $r: \mathbf{A} \to \mathbf{B}$ is an approximable mapping. We must show that $r \in |\mathbf{A} \to \mathbf{B}|$.

Consistency of r. Suppose $X_i r b_i$ for all $i \in \{1, \dots, n\}$ and $X := \bigcup\{X_i : i \in I\} \in \mathrm{Con}_A$ for some $I \subseteq \{1, \dots, n\}$. We must show that $\{b_i : i \in I\} \in \mathrm{Con}_B$. Now from $X_i r b_i$ and $X \vdash_A X_i$ we obtain Xrb_i by 3.1(iii) for any $i \in I$, and hence $\{b_i : i \in I\} \in \mathrm{Con}_B$ by 3.1(i).

Deductive closure of r. Suppose $X_i r b_i$ for all $i \in \{1, \dots, n\}$ and $W := \{(X_1, b_1), \dots, (X_n, b_n)\} \vdash (X, b)$. We must show Xrb. By the definition of $\vdash$ for $\mathbf{A} \to \mathbf{B}$ we have $WX \vdash_B b$, which is $\{b_i : X \vdash_A X_i\} \vdash_B b$. Further by assumption $X_i r b_i$ we know Xrb_i by 3.1(iii) for all i with $X \vdash_A X_i$. Hence Xrb by 3.1(ii). $\square$

4.10 Theorem. *Let $\mathbf{A} = (A, \mathrm{Con}_A, \vdash_A)$ and $\mathbf{B} = (B, \mathrm{Con}_B, \vdash_B)$ be information systems. Then*

$$|\mathbf{A} \to \mathbf{B}| \cong |\mathbf{A}| \to |\mathbf{B}|.$$

The isomorphism pair is

$$\varphi: |\mathbf{A} \to \mathbf{B}| \to (|\mathbf{A}| \to |\mathbf{B}|) \quad \textit{given by} \quad \varphi(s) = |s|,$$

$$\psi: (|\mathbf{A}| \to |\mathbf{B}|) \to |\mathbf{A} \to \mathbf{B}| \quad \textit{given by} \quad \psi(f) = r_f.$$

Proof. $\psi \circ \varphi = \mathrm{id}_{|\mathbf{A} \to \mathbf{B}|}$ and $\varphi \circ \psi = \mathrm{id}_{|\mathbf{A}| \to |\mathbf{B}|}$ has been shown in Theorem 3.2.

φ is monotone. Suppose $s, s' \in |\mathbf{A} \to \mathbf{B}|$ and $s \subseteq s'$. We must show that $|s| \leq |s'|$ in the pointwise order of the continuous functions in $|\mathbf{A}| \to |\mathbf{B}|$. So let $z \in |\mathbf{A}|$. Then we have

$$\begin{aligned} |s|(z) &= \{b \in B : Xsb \text{ for some } X \subseteq^{\text{fin}} z\} \\ &\subseteq \{b \in B : Xs'b \text{ for some } X \subseteq^{\text{fin}} z\} \\ &= |s'|(z). \end{aligned}$$

ψ is monotone. Suppose $f, f': |\mathbf{A}| \to |\mathbf{B}|$ are continuous and $f(z) \subseteq f'(z)$ for all $z \in |\mathbf{A}|$. Then for any $X \in \mathrm{Con}_A$ and $b \in B$ we have

$$\begin{aligned} Xr_f b &\iff b \in f(\overline{X}) \\ &\implies b \in f'(\overline{X}) \\ &\iff Xr_{f'} b. \end{aligned}$$

$\square$

4.11 Corollary. *Suppose D and E are domains with countable bases. Then the set $D \to E$ of all continuous functions from D to E equipped with the pointwise order is again a domain with countable basis.*

Proof. $D \to E \cong |ID| \to |IE| \cong |ID \to IE|$. $\square$

The cpo's together with the continuous functions between them form a cartesian closed category. We only need here the continuity of application, hence we restrict ourselves to a proof of this fact. We first prove the well–known universal property of cartesian products. Another way to put this is to say that a function into a product is continuous if its component functions are continuous. Note that the converse follows from Lemma 2.4 and the continuity of the projections.

4.12 Lemma. *Suppose F, D and E are cpo's. Let $f: F \to D$ and $g: F \to E$ be continuous functions. Then there is a unique continuous function $h: F \to D \times E$ such that $\pi_{\text{left}} \circ h = f$ and $\pi_{\text{right}} \circ h = g$. h is denoted by $f \times g$.*

Proof. Suppose h is such a function. Then for any $x \in F$ $h(x) = (f(x), g(x))$, so h is uniquely determined. For existence, define h by $h(x) = (f(x), g(x))$. Then h satisfies the equations above, and clearly h is monotone. It remains to prove that h is continuous. So let $M \subseteq F$ be directed. Then we have

$$\begin{aligned} h(\bigsqcup M) &= (f(\bigsqcup M), g(\bigsqcup M)) \\ &= (\bigsqcup f(M), \bigsqcup g(M)) \\ &= (\bigsqcup \pi_{\text{left}}(h(M)), \bigsqcup \pi_{\text{right}}(h(M))) \\ &= \bigsqcup h(M) && \text{by Lemma 4.1. } \square \end{aligned}$$

The next thing to do is to show that the sections ι_1^e and ι_2^d of the identity function on $D \times E$ are continuous. By these we mean the functions $\iota_1^e: D \to D \times E$ given by $\iota_1^e(d) = (d, e)$ for any fixed $e \in E$, and $\iota_2^d: E \to D \times E$ given by $\iota_2^d(e) = (d, e)$ for any fixed $d \in D$. Clearly e.g. ι_1^e is monotone. To see that it is continuous, let $M \subseteq D$ be directed. Then clearly $\{(d, e) : d \in M\}$ is directed too, and $\bigsqcup_{d \in M}(d, e) = (\bigsqcup M, e)$.

Now we can show that a function from a product $D \times E$ is continuous if its sections are continuous, or as one might say if it is continuous in each component separately. Note that the converse follows from the continuity of the functions ι_1^e and ι_2^d.

4.13 Lemma. *Suppose D, E and F are cpo's. Then a function $f: D \times E \to F$ is continuous if and only if it is continuous in each component separately, i.e. if all sections $f_1^e: D \to F$ given by $f_1^e(d) = f(d, e)$ and $f_2^d: E \to F$ given by $f_2^d(e) = f(d, e)$ are continuous.*

Proof. If f is continuous, then its sections $f_1^e = f \circ \iota_1^e$ and $f_2^d = f \circ \iota_2^d$ are continuous by Lemma 2.4 and the continuity of the functions ι_1^e and ι_2^d.

Conversely, we first show that f is monotone. So let $(d, e) \leq (d', e')$. Then $f(d, e) = f_1^e(d) \leq f_1^e(d') = f(d', e) = f_2^{d'}(e) \leq f_2^{d'}(e') = f(d', e')$. Now assume that $M \subseteq D \times E$ is directed. We must show $f(\bigsqcup M) = \bigsqcup f(M)$. Clearly $f(\bigsqcup M)$ is an upper bound of $f(M)$. Now suppose u is some upper bound of $f(M)$. We must show $f(\bigsqcup M) \leq u$. Now by Lemma 4.1 $\bigsqcup \pi_{\text{left}}(M)$ and $\bigsqcup \pi_{\text{right}}(M) =: e^*$ exist and we have

$$\begin{aligned} f(\bigsqcup M) &= f(\bigsqcup \pi_{\text{left}}(M), \bigsqcup \pi_{\text{right}}(M)) \\ &= \bigsqcup_{d \in \pi_{\text{left}}(M)} f(d, \bigsqcup \pi_{\text{right}}(M)) && \text{since } f_1^{e^*} \text{ is continuous} \\ &= \bigsqcup_{d \in \pi_{\text{left}}(M)} \bigsqcup_{e \in \pi_{\text{right}}(M)} f(d, e) && \text{since } f_2^d \text{ is continuous.} \end{aligned}$$

Let $d \in \pi_{\text{left}}(M)$ and $e \in \pi_{\text{right}}(M)$. Then $(d, e') \in M$ for some $e' \in E$ and $(d', e) \in M$ for some $d' \in D$. Since M is directed, there is some $(d'', e'') \in M$ with $d, d' \leq d''$ and $e, e' \leq e''$. Hence from the monotonicity of f we get $f(d, e) \leq f(d'', e'') \leq u$. Since this holds for any $e \in \pi_{\text{right}}(M)$, we have $\bigsqcup_{e \in \pi_{\text{right}}(M)} f(d, e) \leq u$, and since this again holds for any $d \in \pi_{\text{left}}(M)$, we get $\bigsqcup_{d \in \pi_{\text{left}}(M)} \bigsqcup_{e \in \pi_{\text{right}}(M)} f(d, e) \leq u$. $\square$.

Now we can show that application is continuous.

4.14 Lemma. *Let D and E be cpo's. Then* $\text{apply}: (D \to E) \times D \to E$ *given by* $\text{apply}(f, d) = f(d)$ *is continuous.*

Proof. By Lemma 4.13 it suffices to prove that apply is continuous in each argument separately. For the second argument this is trivial, and monotonicity in the first argument follows from the pointwise definition of $\leq$ in $D \to E$. So assume $\{f_i : i \in I\} \subseteq D \to E$ is directed. Then we have

$$\begin{aligned} \text{apply}(\bigsqcup_{i \in I} f_i, d) &= (\bigsqcup_{i \in I} f_i)(d) \\ &= \bigsqcup_{i \in I} f_i(d) && \text{by Lemma 4.6} \\ &= \bigsqcup_{i \in I} \text{apply}(f_i, d). && \square \end{aligned}$$

5. Partial continuous functionals

We now construct the partial continuous functionals. First we define for any simple type ϱ an information system $\mathbf{C}_\varrho = (C_\varrho, \mathrm{Con}_\varrho, \vdash_\varrho)$ by letting $\mathbf{C}_{\mathrm{nat}} := \mathbf{N}$, i.e. the information system built from the set of natural numbers as described after Definition 1.1, and

$$\begin{aligned} \mathbf{C}_{\varrho\to\sigma} &:= \mathbf{C}_\varrho \to \mathbf{C}_\sigma, \\ \mathbf{C}_{\varrho\times\sigma} &:= \mathbf{C}'_\varrho \times \mathbf{C}''_\sigma, \end{aligned}$$

where $\mathbf{C}'_\varrho$, $\mathbf{C}''_\sigma$ are variants of $\mathbf{C}_\varrho$, $\mathbf{C}_\sigma$ with disjoint sets of data objects (e.g. $C'_\varrho = \{(0,a) : a \in C_\varrho\}$ and $C''_\sigma = \{(1,a) : a \in C_\sigma\}$).

The information systems $\mathbf{C}_\varrho$ have a rather pleasant property, which amounts to the possibility to locate inconsistencies in two–element sets of data objects; this property has been called coherent by Plotkin in [Plo78], p.210. It is defined as follows. An information system $\mathbf{A} = (A, \mathrm{Con}, \vdash)$ is called *coherent* if for any set $X \subseteq^{\mathrm{fin}} A$ we have

$$X \in \mathrm{Con} \iff (\forall a, b \in X)\{a, b\} \in \mathrm{Con}.$$

Clearly $\mathbf{C}_{\mathrm{nat}} = \mathbf{N}$ is coherent, and furthermore we have

Lemma 5.1. *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be information systems.*
(i) If $\mathbf{B}$ *is coherent, then so is* $\mathbf{A} \to \mathbf{B}$.
(ii) If $\mathbf{A}$ *and* $\mathbf{B}$ *are coherent, then so is* $\mathbf{A} \times \mathbf{B}$.

Proof. Let $\mathbf{A} = (A, \mathrm{Con}_A, \vdash_A)$ and $\mathbf{B} = (B, \mathrm{Con}_B, \vdash_B)$.

(i)Let $\{(X_1, b_1), \dots, (X_n, b_n)\} \subseteq \mathrm{Con}_A \times B$ and assume

$$\forall i, j.1 \le i < j \le n \to \{(X_i, b_i), (X_j, b_j)\} \in \mathrm{Con}. \tag{$*$}$$

We have to show that $\{(X_1, b_1), \dots, (X_n, b_n)\} \in \mathrm{Con}$. So assume $I \subseteq \{1, \dots, n\}$ and $\bigcup_{i\in I} X_i \in \mathrm{Con}_A$. We must show that $\{b_i : i \in I\} \in \mathrm{Con}_B$. Now since $\mathbf{B}$ is coherent by assumption, it suffices to show that $\{b_i, b_j\} \in \mathrm{Con}_B$ for all $i, j \in I$. So let $i, j \in I$. By assumption we have $X_i \cup X_j \in \mathrm{Con}_A$ and hence by $(*)$ and the definition of Con also $\{b_i, b_j\} \in \mathrm{Con}_B$.

(ii) We may assume that A and B are disjoint. Let $a_1, \dots, a_n \in A$ and $b_1, \dots, b_m \in B$ and assume that any two–element subset of $\{a_1, \dots, a_n, b_1, \dots, b_m\}$ is consistent. Then $\{a_1, \dots, a_n\} \in \mathrm{Con}_A$ and $\{b_1, \dots, b_m\} \in \mathrm{Con}_B$ since $\mathbf{A}$ and $\mathbf{B}$ are coherent. Therefore $\{a_1, \dots, a_n, b_1, \dots, b_m\} \in \mathrm{Con}$ by definition of Con. □

Corollary 5.2. *The information systems* $\mathbf{C}_\varrho$ *are all coherent.* □

We now let $\mathrm{Cont}_\varrho := |\mathbf{C}_\varrho|$ be the domain of elements of $|\mathbf{C}_\varrho|$; these elements are called the *partial continuous functionals* of type ϱ. The partial continuous functionals are partial in the sense that the object undefined, i.e. $\bot := \overline{\emptyset}$, is a possible value. We now can introduce easily the total continuous functionals, i.e. those which for total arguments only produce total values; this idea clearly can be turned into a valid inductive definition. We will show that the total functionals are dense in the sense that any finite partial continuous functional (i.e. any $\overline{X}$ with $X \in \mathrm{Con}_\varrho$) can be extended to a total functional.

The set G_ϱ of the *total* functionals of type ϱ is the subset of Cont_ϱ defined by

$$\begin{aligned} G_{\mathrm{nat}} &:= \{z \in \mathrm{Cont}_{\mathrm{nat}} : z \ne \emptyset\} = \{\{n\} : n \in \mathbf{N}\}, \\ G_{\varrho\to\sigma} &:= \{z \in \mathrm{Cont}_{\varrho\to\sigma} : \forall x \in G_\varrho . zx \in G_\sigma\}, \\ G_{\varrho\times\sigma} &= \{z \in \mathrm{Cont}_{\varrho\times\sigma} : z \cap C_\varrho \in G_\varrho \text{ and } z \cap C_\sigma \in G_\sigma\}. \end{aligned}$$

Here we have written zx instead of the more correct $|z|(x)$ (cf. Theorem 4.10). We will continue to do so below.

We now define a relation $\sim_\varrho$ on G_ϱ. For $z, z' \in G_{\mathrm{nat}}$ let $z \sim_{\mathrm{nat}} z' \iff z = z'$. For $z, z' \in G_{\varrho\to\sigma}$ let $z \sim_{\varrho\to\sigma} z' \iff \forall x \in G_\varrho . zx \sim_\sigma z'x$. For $z, z' \in G_{\varrho\times\sigma}$ let $z \sim_{\varrho\times\sigma} z' \iff \pi_{\mathrm{left}} z \sim_\varrho \pi_{\mathrm{left}} z'$ and $\pi_{\mathrm{right}} z \sim_\sigma \pi_{\mathrm{right}} z'$. Clearly $\sim_\varrho$ is an equivalence relation.

We obviously want to know that $\sim_\varrho$ is compatible with application. The only nontrivial part of this argument is to show that $x \sim_\varrho y \implies zx \sim_\sigma zy$. This has first been proved by Yuri Ershov [Ers75], as an application of the density theorem. However, a simpler proof has later been found by Longo and Moggi [LM84], and we present that one. First we need some Lemmata. The first one notes a rather obvious fact on total functionals.

5.3 Lemma. *If $z \in G_\varrho$, $z' \in \mathrm{Cont}_\varrho$ and $z \subseteq z'$, then $z' \in G_\varrho$.*

Proof. By induction on ϱ. For the ground type nat the claim is obvious.

$\varrho \to \sigma$: Assume $z \in G_{\varrho\to\sigma}$ and $z \subseteq z'$. We must show $z' \in G_{\varrho\to\sigma}$. So let $x \in G_\varrho$. We have to show $z'x \in G_\sigma$. But $z'x \supseteq zx \in G_\sigma$, so the claim follows by induction hypothesis.

$\varrho \times \sigma$: The claim follows easily from the monotonicity of the projection functions. □

5.4 Lemma. *For any $z_1, z_2 \in \mathrm{Cont}_{\varrho\to\sigma}$ and $x \in \mathrm{Cont}_\varrho$ we have $(z_1 \cap z_2)x = z_1x \cap z_2x$.*

Proof. By the definition of $|r|$ in Theorem 3.2 we have

$$\begin{aligned}
|z_1 \cap z_2|x &= \{b \in C_\sigma : (\exists X \subseteq^{\text{fin}} x)(X, b) \in z_1 \cap z_2\} \\
&= \{b \in C_\sigma : (\exists X_1 \subseteq^{\text{fin}} x)(X_1, b) \in z_1\} \cap \{b \in C_\sigma : (\exists X_2 \subseteq^{\text{fin}} x)(X_1, b) \in z_2\} \\
&= |z_1|x \cap |z_2|x
\end{aligned}$$

The part $\subseteq$ of the middle equality is obvious. For $\supseteq$, let $X_i \subseteq^{\text{fin}} x$ with $(X_i, b) \in z_i$ be given. Choose $X = X_1 \cup X_2$. Then clearly $(X, b) \in z_i$ (as $\{(X_i, b)\} \vdash (X, b)$ and z_i is deductively closed). □

5.5 Lemma. *For any $z, z' \in G_\varrho$ we have $z \sim_\varrho z' \iff z \cap z' \in G_\varrho$.*

Proof. By induction on ϱ. For the ground type nat the claim is obvious.

$\varrho \to \sigma$:

$$\begin{aligned}
z \sim_{\varrho\to\sigma} z' &\iff \forall x \in G_\varrho . zx \sim_\sigma z'x \\
&\iff \forall x \in G_\varrho . zx \cap z'x \in G_\sigma && \text{by induction hypothesis} \\
&\iff \forall x \in G_\varrho . (z \cap z')x \in G_\sigma && \text{by Lemma 5.4} \\
&\iff z \cap z' \in G_{\varrho\to\sigma}
\end{aligned}$$

$\varrho \times \sigma$:

$$\begin{aligned}
z \sim_{\varrho\times\sigma} z' &\iff \pi_{\text{left}} z \sim_\varrho \pi_{\text{left}} z' \quad \text{and} \quad \pi_{\text{right}} z \sim_\sigma \pi_{\text{right}} z' \\
&\iff (\pi_{\text{left}} z) \cap (\pi_{\text{left}} z') \in G_\varrho \quad \text{and} \quad (\pi_{\text{right}} z) \cap (\pi_{\text{right}} z') \in G_\sigma \\
&\iff \pi_{\text{left}}(z \cap z') \in G_\varrho \quad \text{and} \quad \pi_{\text{right}}(z \cap z') \in G_\sigma \\
&\iff z \cap z' \in G_{\varrho\times\sigma}
\end{aligned}$$

□

5.6 Theorem. *For any $x, y \in G_\varrho$ and $z \in G_{\varrho\to\sigma}$ we have $x \sim_\varrho y \implies zx \sim_\sigma zy$.*

Proof. Since $x \sim_\varrho y$ we have $x \cap y \in G_\varrho$ by Lemma 5.5. Now $zx, zy \supseteq z(x \cap y)$ by Theorem 3.2, and hence $zx \cap zy \in G_\sigma$ by Lemma 5.3. But this implies $zx \sim_\sigma zy$ again by Lemma 5.5. □

We now prove the density theorem, which says that any finite partial continuous functional (i.e. any $\overline{X}$ with $X \in \mathrm{Con}_\varrho$) can be extended to a total functional. This result is essentially due to Kreisel [Kre59]. The proof we give here has been obtained by specialization of a proof for a more general density theorem (for arbitrary f_0–spaces) given by Berger in [Ber90, Ber93] (this proof also appears in [SHGL94]). Berger's proof in turn has been obtained by abstracting some general features from a proof of a density theorem given by Ershov in [Ers74, Ers75].

5.7 Density Theorem. *For any $X \in \mathrm{Con}_\varrho$ we can find an $x \in G_\varrho$ such that $X \subseteq x$.*

Proof. Call a type ϱ *dense* if $\forall X \in \mathrm{Con}_\varrho \exists x \in G_\varrho . X \subseteq x$. Furthermore call a type ϱ *separating* if

$$\forall X_1, X_2 \in \mathrm{Con}_\varrho (X_1 \cup X_2 \notin \mathrm{Con}_\varrho \to \exists \vec{z} \in G . \emptyset \neq \overline{X_1}\vec{z} \neq \overline{X_2}\vec{z} \neq \emptyset).$$

Here $\vec{z} \in G$ means that $\vec{z}$ is a sequence of total z_i or of 0 or 1 such that $X_j\vec{z}$ is of type nat.

We prove by simultaneous induction on ϱ that any type ϱ is dense and separating. This extension of our claim is helpful to get the induction through.

For the ground type nat both claims are obvious.

$\varrho \to \sigma$ is separating: This will follow from the inductive hypotheses that ϱ is dense and σ is separating. So let $W, W' \in \mathrm{Con}_{\varrho\to\sigma}$ such that $W \cup W' \notin \mathrm{Con}_{\varrho\to\sigma}$. Since $\mathbf{C}_\varrho$ is coherent by Corollary 5.2 there are $(X, a) \in W$ and $(X', a') \in W'$ such that $X \cup X' \in \mathrm{Con}_\varrho$ but $\{a, a'\} \notin \mathrm{Con}_\sigma$. Since ϱ is dense we have a $z \in G_\varrho$ such that $X \cup X' \subseteq z$. Hence

$$a \in \overline{W}z \quad \text{and} \quad a' \in \overline{W'}z.$$

Now since σ is separating there are $\vec{z} \in G$ such that

$$\emptyset \neq \overline{\{a\}}\vec{z} \neq \overline{\{a'\}}\vec{z} \neq \emptyset,$$

hence also

$$\emptyset \neq \overline{W}z\vec{z} \neq \overline{W'}z\vec{z} \neq \emptyset.$$

This concludes the proof that $\varrho \to \sigma$ is separating.

$\varrho \to \sigma$ is dense: This will follow from the inductive hypotheses that ϱ is separating and σ is dense. So let $W = \{(X_i, a_i) : i \in I\} \in \mathrm{Con}_{\varrho\to\sigma}$. Consider i, j such that $\{a_i, a_j\} \notin \mathrm{Con}_\sigma$. Then $X_i \cup X_j \notin \mathrm{Con}_\varrho$. Since ϱ is separating, there are $\vec{z}_{ij} \in G$ and k_{ij}, l_{ij} such that

$$\{k_{ij}\} = \overline{X_i}\vec{z}_{ij} \neq \overline{X_j}\vec{z}_{ij} = \{l_{ij}\}.$$

We clearly may assume that $\vec{z}_{ij} = \vec{z}_{ji}$ and $(k_{ij}, l_{ij}) = (l_{ji}, k_{ji})$.

Now define for any $X \in \mathrm{Con}_\varrho$ a set I_X of indices $i \in I$ such that 'X behaves as X_i with respect to the $\vec{z}_{ij}$'. More precisely, let

$$I_X := \{i \in I : \forall j.\{a_i, a_j\} \notin \mathrm{Con}_\sigma \to \overline{X}\vec{z}_{ij} = \{k_{ij}\}\}.$$

We first show that

$$\{a_i : i \in I_X\} \in \mathrm{Con}_\sigma. \tag{1}$$

Since $\mathbf{C}_\sigma$ is coherent it suffices to show that $\{a_i, a_j\} \in \mathrm{Con}_\sigma$ for all $i, j \in I_X$. So let $i, j \in I_X$ and assume $\{a_i, a_j\} \notin \mathrm{Con}_\sigma$. Then we have

$$\overline{X}\vec{z}_{ij} = \{k_{ij}\} \quad \text{as } i \in I_X$$

and

$$\overline{X}\vec{z}_{ji} = \{k_{ji}\} \quad \text{as } j \in I_X,$$

and because of $\vec{z}_{ij} = \vec{z}_{ji}$ and $k_{ij} \neq l_{ij} = k_{ji}$ we could conclude that $\overline{X}\vec{z}_{ij}$ would be inconsistent. This contradiction proves $\{a_i, a_j\} \in \mathrm{Con}_\sigma$ and hence (1).

Since (1) holds and σ is dense by induction hypothesis, we can find a $y_{I_X} \in G_\sigma$ such that $a_i \in y_{I_X}$ for all $i \in I_X$. Now define a relation $r \subseteq \mathrm{Con}_\varrho \times C_\sigma$ by

$$Xra :\Longleftrightarrow \begin{cases} a \in y_{I_X}, & \text{if } \overline{X} \text{ is defined on all } \vec{z}_{ij}; \\ \{a_i : i \in I_X\} \vdash_\sigma a, & \text{otherwise,} \end{cases}$$

where 'x is defined on $\vec{z}$' means $x\vec{z} \neq \emptyset$. We will show that $r \in G_{\varrho\to\sigma}$ and $W \subseteq r$.

For $W \subseteq r$ we have to show $X_i r a_i$ for all $i \in I$. But this holds, since clearly $i \in I_{X_i}$ and hence $\{a_j : j \in I_{X_i}\} \vdash_\sigma a_i$ and also $a_i \in y_{I_{X_i}}$.

We now show that r is an approximable mapping; by Lemma 4.9 this means $r \in \mathrm{Cont}_{\varrho\to\sigma}$. To prove this we have to verify conditions (i) to (iii) from Definition 3.1.

(i) $Xrb_1, \dots, Xrb_n \implies \{b_1, \dots, b_n\} \in \mathrm{Con}_\sigma$. If $\overline{X}$ is defined on all $\vec{z}_{ij}$ the claim follows from the consistency of y_{I_X}. If not, the claim follows from the properties of $\vdash_\sigma$.

(ii) $Xrb_1, \dots, Xrb_n, \{b_1, \dots, b_n\} \vdash_\sigma a \implies Xra$. If $\overline{X}$ is defined on all $\vec{z}_{ij}$ the claim follows from the deductive closure of y_{I_X}. If not, the claim follows from the properties of $\vdash_\sigma$.

(iii) $X \vdash_\varrho X', X'ra \implies Xra$. Case 1. $\overline{X'}$ is defined on all $\vec{z}_{ij}$. Then also $\overline{X}$ is defined on all $\vec{z}_{ij}$. From $X'ra$ we get $a \in y_{I_{X'}}$. We have to show that $a \in y_{I_X}$. Now since X and X' are defined on all $\vec{z}_{ij}$ and $X \vdash_\varrho X'$, they must have the same values on the $\vec{z}_{ij}$, hence $I_{X'} = I_X$ and therefore $y_{I_{X'}} = y_{I_X}$. Case 2. Otherwise. From $X'ra$ we get $\{a_i : i \in I_{X'}\} \vdash_\sigma a$. Now from $X \vdash_\varrho X'$ we can conclude $I_{X'} \subseteq I_X$, by the definition of I_X. Hence $\{a_i : i \in I_X\} \vdash_\sigma a$, and also $a \in y_{I_X}$ (since $a_i \in y_{I_X}$ for all $i \in I_X$, and y_{I_X} is deductively closed). Therefore Xra.

This concludes the proof that r is an approximable mapping.

It remains to show that $r \in G_{\varrho\to\sigma}$. So let $x \in G_\varrho$. We must show

$$|r|(x) = \{a \in C_\sigma : \exists X \subseteq^{\text{fin}} x. Xra\} \in G_\sigma.$$

Now for any $i, j \in I$ we have $x\vec{z}_{ij} \neq \emptyset$, hence there is some $X_{ij} \subseteq^{\text{fin}} x$ such that $\overline{X_{ij}}\vec{z}_{ij} \neq \emptyset$. Let $X \subseteq^{\text{fin}} x$ be the union of all the X_{ij}. Then clearly $\overline{X}$ is defined on all $\vec{z}_{ij}$ and hence Xra for all $a \in y_{I_X}$. Therefore $y_{I_X} \subseteq |r|(x)$ and hence $|r|(x) \in G_\sigma$ by Lemma 5.3.

This concludes the proof that $\varrho \to \sigma$ is dense.

$\varrho \times \sigma$ is dense and separating: This follows easily from the induction hypotheses that ϱ and σ are dense and separating. $\square$

As an application of the Density Theorem we prove a choice principle for total continuous functionals. This result was first proved by Kreisel in [Kre59]; our proof essentially follows Berger [Ber93].

5.8 Theorem (Choice Principle for Total Continuous Functionals). *We can construct an object $r \in \text{Cont}_{(\varrho\to\sigma\to\text{nat})\to\varrho\to\sigma}$ such that for any $F \in G_{\varrho\to\sigma\to\text{nat}}$ satisfying*

$$(\forall x \in G_\varrho)(\exists y \in G_\sigma) Fxy = \{0\}$$

we have $rF \in G_{\varrho\to\sigma}$ and

$$(\forall x \in G_\varrho) Fx(rFx) = \{0\}.$$

Proof. Let $Y_0, Y_1, Y_2, \dots$ be an enumeration of Con_σ. By the Density Theorem 5.7 we can find for any Y_n a $y_n \in G_\sigma$ such that $Y_n \subseteq y_n$. Define for any $W \in \text{Con}_{\varrho\to\sigma\to\text{nat}}$, $X \in \text{Con}_\varrho$ and $a \in C_\sigma$

$$Wr(X,a) \iff (\exists m, \vec{k}. (\forall i < m)\overline{W}\,\overline{X} y_i = \{k_i + 1\} \wedge \overline{W}\,\overline{X} y_m = \{0\} \wedge a \in y_m) \vee \emptyset \vdash_\sigma a.$$

We first show that r is an approximable mapping. To prove this we have to verify conditions (i) to (iii) from Definition 3.1.

(i) $Wr(X_1, a_1), \dots, Wr(X_n, a_n) \implies \{(X_1, a_1), \dots, (X_n, a_n)\} \in \text{Con}_{\varrho\to\sigma}$. Assume the premise and $X := \bigcup_{i\in I} X_i \in \text{Con}_\varrho$. We must show that $\{a_i : i \in I\} \in \text{Con}_\sigma$. If $\emptyset \vdash_\sigma a_i$ for all $i \in I$ we are done. Otherwise for all $i \in I$ such that $\emptyset \not\vdash_\sigma a_i$ the numbers m_i in the definition of $Wr(X_i, a_i)$ are all the same, $= m$ say. Hence $\{a_i : i \in I\} \subseteq y_m$, and the claim follows from the consistency of y_m.

(ii) $Wr(X_1, a_1), \dots, Wr(X_n, a_n), \{(X_1, a_1), \dots, (X_n, a_n)\} \vdash_{\varrho\to\sigma} (X, a) \implies Wr(X, a)$. Assume the premise and $I := \{i : X \vdash_\varrho X_i\}$. We know that $\{a_i : i \in I\} \vdash_\sigma a$. If $\emptyset \vdash_\sigma a_i$ for all $i \in I$ we are done. Otherwise for all $i \in I$ such that $\emptyset \not\vdash_\sigma a_i$ the numbers m_i in the definition of $Wr(X_i, a_i)$ are all the same, $= m$ say. Hence $\{a_i : i \in I\} \subseteq y_m$, and the claim follows from the deductive closure of y_m.

(iii) $W \vdash_{\varrho\to\sigma\to\text{nat}} W', W'r(X, a) \implies Wr(X, a)$. This clearly follows from the definition of r; the m from $W'r(X, a)$ can be used for $Wr(X, a)$.

We finally show that for all $F \in G_{\varrho\to\sigma\to\text{nat}}$ satisfying

$$(\forall x \in G_\varrho)(\exists y \in G_\sigma) Fxy = \{0\}$$

and for all $x \in G_\varrho$ we have $rFx \in G_\sigma$ and $Fx(rFx) = \{0\}$. So let F and x with these properties be given. By assumption there is a $y \in G_\sigma$ such that $Fxy = \{0\}$. Hence by the definition of application there is a $Y_n \in \text{Con}_\sigma$ such that $Fx\overline{Y_n} = \{0\}$. Since $Y_n \subseteq y_n$ we also have $Fxy_n = \{0\}$. Clearly we may assume here that n is minimal with this property, i.e. that

$$Fxy_0 = \{k_0 + 1\}, \dots, Fxy_{n-1} = \{k_{n-1} + 1\}.$$

We show that $rFx \supseteq y_n$ or more precisely $||r|(F)|(x) \supseteq y_n$; this suffices by Lemma 5.3. Recall that

$$|r|(F) = \{(X,a) \in \mathrm{Con}_\varrho \times C_\sigma : (\exists W \subseteq^{\mathrm{fin}} F) W r(X,a)\}$$

and

$$\begin{aligned} ||r|(F)|(x) &= \{a \in C_\sigma : (\exists X \subseteq^{\mathrm{fin}} x)(X,a) \in |r|(F)\} \\ &= \{a \in C_\sigma : (\exists X \subseteq^{\mathrm{fin}} x)(\exists W \subseteq^{\mathrm{fin}} F) W r(X,a)\}. \end{aligned}$$

Now let $a \in y_n$. By the choice of n we then get $X \subseteq^{\mathrm{fin}} x$ and $W \subseteq^{\mathrm{fin}} F$ such that

$$(\forall i < n)\overline{W}\,\overline{X} y_i = \{k_i + 1\} \qquad \text{and} \qquad \overline{W}\,\overline{X} y_n = \{0\}.$$

Therefore $Wr(X,a)$ and hence $a \in ||r|(F)|(x)$. □

We finally comment of the notion of effectivity in this context. In particular we want to define what it means for a partial continuous functional of an arbitrary simple type ϱ to be computable.

An information system $(A, \mathrm{Con}, \vdash)$ is called *effective* if the (countable) set A, the set Con and the relation $\vdash$ are all decidable. Since decidability always means decidability relative to a given set we must assume here that the data objects are taken from a fixed given set, e.g. the set of natural numbers or the set of all strings over some finite alphabet. It is easy to see that all the operations on information systems introduced in section 4 preserve effectivity.

In an effective information system it makes sense to talk about a recursively enumerable (r.e.) ideal, since any ideal is a set of data objects. The *computable* elements of an effective information system are defined to be its r.e. ideals. Clearly application does not carry us out of the realm of computable elements, i.e. if $r \in |\mathbf{A} \to \mathbf{B}|$ and $x \in |\mathbf{B}|$ are computable, then so is rx (or more precisely $|r|(x)$). Also we can compute an r.e. index of $|r|x$ from those of r and x.

Let us now look somewhat closer at our particular information systems $\mathbf{C}_\varrho$, whose elements are the partial continuous functionals. A *computable* functional of type ϱ is defined to be a computable element of $\mathbf{C}_\varrho$. The set of all computable functionals of type ϱ is denoted by Comp_ϱ.

5.9 Effective Density Theorem. *For any $X \in \mathrm{Con}_\varrho$ we can find effectively an $x \in G_\varrho \cap \mathrm{Comp}_\varrho$ such that $X \subseteq x$.*

Proof. By inspection of the proof of the Density Theorem 5.7. To see that r (in the proof that $\varrho \to \sigma$ is dense) is effective one needs that

– $\{a_i : i \in I_X\} \vdash_\sigma a$ implies $a \in y_{I_X}$ for all X and a, since by definition $a_i \in y_{I_X}$ for all $i \in I_X$. Hence

$$Xra \iff \{a_i : i \in I_X\} \vdash_\sigma a \quad \text{or} \quad (a \in y_{I_X} \quad \text{and} \quad \overline{X} \text{ is defined on all } \vec{z_{ij}}).$$

– If $\overline{X}$ is defined on all $\vec{z_{ij}}$, one can actually compute I_X (and not only an enumeration procedure for I_X). □

5.10 Theorem (Effective Choice Principle). *There is an $r \in \mathrm{Comp}_{(\varrho\to\sigma\to\mathrm{nat})\to\varrho\to\sigma}$ such that for any $F \in G_{\varrho\to\sigma\to\mathrm{nat}}$ satisfying*

$$(\forall x \in G_\varrho)(\exists y \in G_\sigma) Fxy = \{0\}$$

we have $rF \in G_{\varrho\to\sigma}$ and

$$(\forall x \in G_\varrho) Fx(rFx) = \{0\}.$$

Proof. Immediate from the proof of the choice principle for total continuous functionals in 5.8. □

References

[Ber90] Ulrich Berger. *Totale Objekte und Mengen in der Bereichstheorie.* PhD thesis, Mathematisches Institut der Universität München, 1990.

[Ber93] Ulrich Berger. Total sets and objects in domain theory. *Annals of Pure and Applied Logic*, 60:91–117, 1993.

[Ers74] Yuri L. Ershov. The model G of the theory BR. *Soviet Math. Doklady*, 15(4):1158–1160, 1974.

[Ers75] Yuri L. Ershov. Theorie der Numerierungen II. *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik*, 21:473–584, 1975.

[Ers77] Yuri L. Ershov. Model C of partial continuous functionals. In R. Gandy and M. Hyland, editors, *Logic Colloquium 1976*, pages 455–467. North–Holland, Amsterdam, 1977.

[Kle59] Stephen C. Kleene. Countable functionals. In A. Heyting, editor, *Constructivity in Mathematics*, pages 81–100. North–Holland, Amsterdam, 1959.

[Kre59] Georg Kreisel. Interpretation of analysis by means of constructive functionals of finite types. In A. Heyting, editor, *Constructivity in Mathematics*, pages 101–128. North–Holland, Amsterdam, 1959.

[LM84] G. Longo and E. Moggi. The hereditary partial effective functionals and recursion theory in higher types. *The Journal of Symbolic Logic*, 49(4):1319–1332, December 1984.

[LW84] K.G. Larsen and G. Winskel. Using information systems to solve recursive domain equations effectively. In *Proceedings of the Conference on Abstract Datatypes, Sophia–Antipolis, France*, pages 109–129, Berlin, Heidelberg, New York, 1984. Springer. Lecture Notes in Computer Science, Vol. 173.

[Nor93] Dag Normann. Closing the gap between the continuous functionals and recursion in ${}^3E$. Submitted to the proceedings of the Sacks conference, MIT, Cambridge, 1993.

[Pla66] Richard A. Platek. *Foundations of recursion theory.* PhD thesis, Department of Mathematics, Stanford University, 1966.

[Plo77] Gordon D. Plotkin. LCF considered as a programming language. *Theoretical Computer Science*, 5:223–255, 1977.

[Plo78] Gordon D. Plotkin. $\mathbf{T}^\omega$ as a universal domain. *Journal of Computer and System Sciences*, 17:209–236, 1978.

[Ros87] A.W. Roscoe. Notes on domain theory, 1987. Unpublished notes, Oxford University.

[Sch91] Helmut Schwichtenberg. Primitive recursion on the partial continuous functionals. In M. Broy, editor, *Informatik und Mathematik*, pages 251–269. Springer, Berlin, 1991.

[Sco70] Dana Scott. Outline of a mathematical theory of computation. Technical Monograph PRG–2, Oxford University Computing Laboratory, 1970.

[Sco82] Dana Scott. Domains for denotational semantics. In E. Nielsen and E.M. Schmidt, editors, *Automata, Languages and Programming*, volume 140 of *Lecture Notes in Computer Science*, pages 577–613. Springer Verlag, Berlin, Heidelberg, New York, 1982. A corrected and expanded version of a paper prepared for ICALP'82, Aarhus, Denmark.

[SHGL94] Viggo Stoltenberg-Hansen, Edward Griffor, and Ingrid Lindström. *Mathematical Theory of Domains.* Cambridge Tracts in Theoretical Computer Science. Cambridge University Press, 1994.