EMBEDDING CLASSICAL IN MINIMAL IMPLICATIONAL LOGIC

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ABSTRACT. Consider the problem which set V of propositional variables suffices for $\mathrm{Stab}_V \vdash_i A$ whenever $\vdash_c A$, where $\mathrm{Stab}_V := \{ \neg\neg P \to P \mid P \in V \}$, and $\vdash_c$ and $\vdash_i$ denote derivability in classical and intuitionistic implicational logic, respectively. We give a direct proof that stability for the final propositional variable of the (implicational) formula A is sufficient; as a corollary one obtains Glivenko's theorem. Conversely, using Glivenko's theorem one can give an alternative proof of our result. As an alternative to stability we then consider the Peirce formula $\mathrm{Peirce}_{Q,P} := ((Q \to P) \to Q) \to Q$. It is an easy consequence of the result above that adding a single instance of the Peirce formula suffices to move from classical to intuitionistic derivability. Finally we consider the question whether one could do the same for minimal logic. Given a classical derivation of a propositional formula not involving $\bot$, which instances of the Peirce formula suffice as additional premises to ensure derivability in minimal logic? We define a set of such Peirce formulas, and show that in general an unbounded number of them is necessary.

Keywords: Minimal implicational logic, classical implicational logic, intuitionistic implicational logic, stability, Peirce formula

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1. Introduction

The formulas $A, B, \dots$ of implicational (propositional) logic are built from propositional variables $P, Q, \dots$ by implication $\to$ alone. Let $\bot$ (falsity) and $*$ be distinguished propositional variables. We define $\neg A := A \to \bot$ and $\neg_* A := A \to *$. Let $\vdash_c$ and $\vdash_i$ denote classical and intuitionistic derivability, respectively. By definition[1], $\vdash_c A$ means $\mathrm{Stab}_{\mathcal{V}(A)} \vdash A$ and $\vdash_i A$ means $\mathrm{Efq}_{\mathcal{V}(A)} \vdash A$, where $\vdash$ denotes derivability in minimal logic, $\mathrm{Stab}_V := \{ \neg\neg P \to P \mid P \in V \}$ and $\mathrm{Efq}_V := \{ \bot \to P \mid P \in V \}$, and $\mathcal{V}(A)$ is the set of propositional variables in the formula A.

We consider the problem which set V of propositional variables suffices for $\mathrm{Stab}_V \vdash_i A$ whenever $\vdash_c A$, and give a direct proof that stability for the final propositional variable of the (implicational) formula A is sufficient.

In [2] a similar problem was solved, where instead of Stab_V decidability assumptions $\Pi_V := \{ P \vee \neg P \mid P \in V \}$ were used. Our proof method is similar to the one employed in [2].

From the result above one easily obtains Glivenko's theorem. Conversely, using Glivenko's theorem one can give an easy alternative proof of our result.

Using stability rather than decidability assumptions is of interest because it allows to stay in the implicational fragment of minimal logic (and hence

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[1] This holds since the constructive $\vee, \exists$ are not in our language; cf. e.g. [3, 1.1.8].

in the pure typed lambda calculus). For example, from $\vdash_c ((Q \to P) \to Q) \to Q$ we obtain

$$\vdash (\bot \to P) \to (\neg\neg Q \to Q) \to ((Q \to P) \to Q) \to Q.$$

As an alternative to stability we consider the Peirce formula $\mathrm{Peirce}_{Q,P} := ((Q \to P) \to Q) \to Q$. It is an easy consequence of the theorem that adding a single instance of the Peirce formula suffices to move from classical to intuitionistic derivability. In fact, $\vdash_c A$ implies $\mathrm{Peirce}_{P,\bot} \vdash_i A$ with P the final conclusion of A. At this point it is a natural question whether one could do the same for minimal logic. Given a classical derivation of a propositional formula not involving $\bot$, which instances of Peirce formulas suffice as additional premises to ensure derivability in minimal logic? We define a set of such Peirce formulas, and show by means of an example that in general an unbounded number of them is necessary.

2. Intuitionistic logic and stability

We work with Gentzen's natural deduction calculus; see [4] for its definition and the necessary background. We will use the following properties of the operators $\neg$ and $\neg_*$.

$$\vdash (\neg\neg * \to *) \to \neg_*\neg A \to \neg_*\neg_* A, \tag{1}$$

$$\vdash (\bot \to B) \to (\neg_*\neg A \to \neg_*\neg_* B) \to \neg_*\neg_*(A \to B). \tag{2}$$

Proof of (1). From $u\colon \neg_*\neg A$, $v\colon \neg_* A$ and $w\colon \neg\neg * \to *$ we obtain

$$
\begin{prooftree}
\AxiomC{$w\colon \neg\neg * \to *$}
\AxiomC{$y\colon \neg *$}
\AxiomC{$u\colon \neg_*\neg A$}
\AxiomC{$y\colon \neg *$}
\AxiomC{$v\colon \neg_* A$}
\AxiomC{$x\colon A$}
\BinaryInfC{$*$}
\BinaryInfC{$\bot$}
\RightLabel{$\to^+ x$}
\UnaryInfC{$A \to \bot$}
\BinaryInfC{$*$}
\BinaryInfC{$\bot$}
\RightLabel{$\to^+ y$}
\UnaryInfC{$\neg\neg *$}
\BinaryInfC{$*$}
\end{prooftree}
$$

as required. □

Proof of (2). From $v\colon \neg_*(A \to B)$ and $w\colon \bot \to B$ we obtain $M(v, w) :=$

$$
\begin{prooftree}
\AxiomC{$v\colon \neg_*(A \to B)$}
\AxiomC{$w\colon \bot \to B$}
\AxiomC{$y\colon \neg A$}
\AxiomC{$x\colon A$}
\BinaryInfC{$\bot$}
\BinaryInfC{B}
\RightLabel{$\to^+ x$}
\UnaryInfC{$A \to B$}
\BinaryInfC{$*$}
\RightLabel{$\to^+ y$}
\UnaryInfC{$(A \to \bot) \to *$}
\end{prooftree}
$$

and again from $v\colon \neg_*(A \to B)$ we get $N(v) :=$

$$
\begin{prooftree}
\AxiomC{$v\colon \neg_*(A \to B)$}
\AxiomC{$z\colon B$}
\UnaryInfC{$A \to B$}
\BinaryInfC{$*$}
\RightLabel{$\to^+ z$}
\UnaryInfC{$B \to *$}
\end{prooftree}
$$

From $u\colon \neg_*\neg A \to \neg_*\neg_* B$ and $M(v,w)$, $N(v)$ we finally obtain

$$
\begin{prooftree}
\AxiomC{$u\colon \neg_*\neg A \to \neg_*\neg_* B$}
\AxiomC{$M(v,w)\colon (A \to \bot) \to *$}
\BinaryInfC{$\neg_*\neg_* B$}
\AxiomC{$N(v)\colon B \to *$}
\BinaryInfC{$*$}
\RightLabel{$\to^+ v$}
\UnaryInfC{$\neg_*\neg_*(A \to B)$}
\RightLabel{$\to^+ u, w$}
\UnaryInfC{$(\bot \to B) \to (\neg_*\neg A \to \neg_*\neg_* B) \to \neg_*\neg_*(A \to B)$}
\end{prooftree}
$$

which was to be shown. □

Proposition 2.1. (a) *Assume* $\Gamma \vdash_c A$. *Then* $\mathrm{Stab}_*, \neg_*\neg\Gamma \vdash_i \neg_*\neg_* A$, *where* $\neg_*\neg\Gamma := \{\, \neg_*\neg A \mid A \in \Gamma \,\}$.
(b) $\Gamma \vdash_c A$, *then* $\mathrm{Stab}_*, \Gamma \vdash_i \neg_*\neg_* A$.

Proof. (a) By induction on $\Gamma \vdash_c A$.

Case Ax. Since our only axiom is stability $\neg\neg A \to A$ we must prove $\mathrm{Stab}_* \vdash_i \neg_*\neg_*(\neg\neg A \to A)$. From $u\colon \neg_*(\neg\neg A \to A)$, $v\colon \bot \to A$ and an auxiliary assumption $y\colon \neg *$ we obtain[2] $M(u,v,y) :=$

$$
\begin{prooftree}
\AxiomC{$v\colon \bot \to A$}
\AxiomC{$z\colon \neg\neg A$}
\AxiomC{$y\colon \neg *$}
\AxiomC{$u\colon \neg_*(\neg\neg A \to A)$}
\AxiomC{$x\colon A$}
\UnaryInfC{$\neg\neg A \to A$}
\BinaryInfC{$*$}
\BinaryInfC{$\bot$}
\RightLabel{$\to^+ x$}
\UnaryInfC{$\neg A$}
\BinaryInfC{$\bot$}
\BinaryInfC{A}
\RightLabel{$\to^+ z$}
\UnaryInfC{$\neg\neg A \to A$}
\end{prooftree}
$$

Again using u we finally obtain

$$
\begin{prooftree}
\AxiomC{$w\colon \neg\neg * \to *$}
\AxiomC{$y\colon \neg *$}
\AxiomC{$u\colon \neg_*(\neg\neg A \to A)$}
\AxiomC{$M(u,v,y)\colon \neg\neg A \to A$}
\BinaryInfC{$*$}
\BinaryInfC{$\bot$}
\RightLabel{$\to^+ y$}
\UnaryInfC{$\neg\neg *$}
\BinaryInfC{$*$}
\end{prooftree}
$$

Case Assumption. We must show $\mathrm{Stab}_*, \neg_*\neg A \vdash_i \neg_*\neg_* A$, which follows from (1) above.

Case $\to^+$.

$$
\begin{prooftree}
\AxiomC{$[u\colon A]$}
\noLine
\UnaryInfC{$\mid M$}
\noLine
\UnaryInfC{B}
\RightLabel{$\to^+ u$}
\UnaryInfC{$A \to B$}
\end{prooftree}
$$

By induction hypothesis

$$\mathrm{Stab}_*, \neg_*\neg\Gamma, \neg_*\neg A \vdash_i \neg_*\neg_* B.$$

The claim $\mathrm{Stab}_*, \neg_*\neg\Gamma \vdash_i \neg_*\neg_*(A \to B)$ follows from (2) above.

Case $\to^-$.

$$
\begin{prooftree}
\AxiomC{$\mid M$}
\noLine
\UnaryInfC{$A \to B$}
\AxiomC{$\mid N$}
\noLine
\UnaryInfC{A}
\RightLabel{$\to^-$}
\BinaryInfC{B}
\end{prooftree}
$$

By induction hypothesis

$$\mathrm{Stab}_*, \neg_*\neg\Gamma \vdash_i \neg_*\neg_*(A \to B),$$

[2] It is easiest to find such a proof with the help of a proof assistant; we have used Minlog `http://www.minlog-system.de`.

$$\mathrm{Stab}_*, \neg_*\neg\Gamma \vdash_i \neg_*\neg_* A.$$

The claim $\mathrm{Stab}_*, \neg_*\neg\Gamma \vdash_i \neg_*\neg_* B$ follows from

$$\neg_*\neg_*(A \to B) \to \neg_*\neg_* A \to \neg_*\neg_* B$$

which can be proved easily (as for $\neg$).

(b) Note that $\vdash (\bot \to *) \to A \to \neg_*\neg A$, and $\mathrm{Stab}_* \vdash \bot \to *$. □

Theorem 2.2. *$\vdash_c A$ implies $\mathrm{Stab}_P \vdash_i A$ for P the final conclusion of A.*

Proof. Let $A = A_1 \to \cdots \to A_n \to P$ and $\Gamma = \{A_1, \dots, A_n\}$. Then by Proposition 2.1(b) we have $\mathrm{Stab}_*, \Gamma \vdash_i \neg_*\neg_* P$. Substituting P for $*$ gives $\mathrm{Stab}_P, \Gamma \vdash_i (P \to P) \to P$ and hence the claim. □

Glivenko's theorem [1] says the every negation proved classically can also be proved intuitionistically. Theorem 2.2 above provides an easy proof of Glivenko's theorem for implicational logic. For $A_1 \to \cdots \to A_n \to B$ we write $\vec{A} \to B$.

Corollary 2.3 (Glivenko)**.** *If $\Gamma \vdash_c \bot$, then $\Gamma \vdash_i \bot$.*

Proof. By Theorem 2.2 $\vdash_c \vec{A} \to \bot$ implies $\mathrm{Stab}_\bot \vdash_i \vec{A} \to \bot$. Now observe that $\mathrm{Stab}_\bot$ is $((\bot \to \bot) \to \bot) \to \bot$ and hence easy to prove. □

In fact, there is an easy alternative proof of Theorem 2.2 from Glivenko's theorem, as follows. Suppose $\vdash_c \vec{A} \to P$. Then also $\vdash_c \neg\neg(\vec{A} \to P)$ and hence by Glivenko's theorem $\vdash_i \neg\neg(\vec{A} \to P)$. Since $\vdash \neg\neg(A \to B) \to A \to \neg\neg B$ we then have $\vdash_i \vec{A} \to \neg\neg P$ and therefore $\mathrm{Stab}_P \vdash_i \vec{A} \to P$.

3. Minimal logic and Peirce

As an alternative to stability we consider the Peirce formula $\mathrm{Peirce}_{Q,P} := ((Q \to P) \to Q) \to Q$. It is an easy consequence of Theorem 2.2 that adding a single instance of the Peirce formula suffices to move from classical to intuitionistic derivability.

Corollary 3.1. *$\vdash_c A$ implies $\mathrm{Peirce}_{P,\bot} \vdash_i A$ (P final conclusion of A).*

Proof. This follows from $\mathrm{Peirce}_{P,\bot} \vdash (\bot \to P) \to \mathrm{Stab}_P$:

$$\dfrac{\dfrac{((P \to \bot) \to P) \to P \qquad \dfrac{\dfrac{\bot \to P \qquad \dfrac{u\colon (P \to \bot) \to \bot \qquad v\colon P \to \bot}{\bot}}{P}}{(P \to \bot) \to P}\,{\to^+} v}{P}}{((P \to \bot) \to \bot) \to P}\,{\to^+} u$$

□

What can be said if we move to minimal logic? Given a classical derivation of a propositional formula not involving $\bot$. we show that finitely many Peirce formulas as addtional premises suffice to obtain a proof in minimal logic. To indicate that we now work in minimal logic we use $*$ rather than $\bot$.

Lemma 3.2 (Peirce suffices for the final atom)**.**

$$\vdash (((* \to B) \to *) \to *) \to ((* \to A \to B) \to *) \to *$$

or in abbreviated notation

$$\vdash \mathrm{Peirce}_{*,B} \to \mathrm{Peirce}_{*,A\to B}.$$

Proof.

$$\dfrac{((* \to B) \to *) \to * \qquad \dfrac{\dfrac{(* \to A \to B) \to * \qquad \dfrac{\dfrac{\dfrac{u\colon * \to B \qquad v\colon *}{B}}{A \to B}\,\to^+ w}{* \to A \to B}\,\to^+ v}{*}}{(* \to B) \to *}\,\to^+ u}{*}$$

□

It is easy to see that $\vdash \neg_*\neg_*(A \to B) \to A \to \neg_*\neg_* B$. However, the converse requires Peirce:

Lemma 3.3 (DNS$_\to$, double negation shift for $\to$)**.**

$$\vdash (((* \to B) \to *) \to *) \to (A \to (B \to *) \to *) \to ((A \to B) \to *) \to *$$

or in abbreviated notation

$$\vdash \mathrm{Peirce}_{*,B} \to (A \to \neg_*\neg_* B) \to \neg_*\neg_*(A \to B).$$

Proof.

$$\dfrac{\mathrm{Peirce}_{*,B} \qquad \dfrac{\dfrac{(A\to B)\to * \qquad \dfrac{\dfrac{u\colon\ * \to B \qquad \dfrac{\dfrac{A\to(B\to *)\to * \quad v\colon A}{(B\to *)\to *} \qquad \dfrac{\dfrac{(A\to B)\to * \quad \dfrac{w\colon B}{A\to B}}{*}}{B\to *}\,\to^+ w}{*}}{B}}{A\to B}\,\to^+ v}{*}}{(*\to B)\to *}\,\to^+ u}{*}$$

□

For the next proposition it will be convenient to work with the sequent calculus **G3cp**; we refer to [4] for its definition and the necessary background. Let Γ, Δ denote multisets of implicational formulas. By induction on derivations $\mathcal{D}\colon \Gamma \Rightarrow \Delta$ in **G3cp** we define a set $\Pi(\mathcal{D})$ of formulas. $\Pi(\mathcal{D})$ will be the set of all Peirce formulas $\mathrm{Peirce}_{*,P}$ for P the final conclusion of a positive implication in $\Gamma \Rightarrow \Delta$, plus possibly (depending on which axioms appear in $\mathcal{D}$) the formula $\bot \to *$.

Cases Ax, L$\bot$. We can assume that Γ and Δ are atomic. If $\Gamma \cap \Delta = \emptyset$ let $\Pi(\mathcal{D}) := \{\bot \to *\}$, and $:= \emptyset$ otherwise.

Case L$\to$. Then $\mathcal{D}$ ends with

$$\dfrac{\begin{matrix} \mid \mathcal{D}_1 \\ \Gamma \Rightarrow \Delta, A \end{matrix} \qquad \begin{matrix} \mid \mathcal{D}_2 \\ B, \Gamma \Rightarrow \Delta \end{matrix}}{A \to B, \Gamma \Rightarrow \Delta}\,\mathrm{L}\!\to$$

Let $\Pi(\mathcal{D}) := \Pi(\mathcal{D}_1) \cup \Pi(\mathcal{D}_2)$.

Case R$\to$. Then $\mathcal{D}$ ends with

$$\dfrac{\begin{array}{c} \mid \mathcal{D}_1 \\ A, \Gamma \Rightarrow \Delta, B \end{array}}{\Gamma \Rightarrow \Delta, A \to B}\ \mathrm{R}{\to}$$

Let $\Pi(\mathcal{D}) := \Pi(\mathcal{D}_1) \cup \{\mathrm{Peirce}_{*,P}\}$ for P the final conclusion of B.

By $\vdash \Gamma \Rightarrow A$ we denote derivability in **G3mp**.

Proposition 3.4. *Let* $\mathcal{D}\colon \Gamma \Rightarrow \Delta$ *in* **G3cp**. *Then* $\vdash \Pi(\mathcal{D}), \Gamma, \neg_* \Delta \Rightarrow *$.

Proof. By induction on the derivation $\mathcal{D}$.

Case Ax. Then $\mathcal{D}\colon P, \Gamma \Rightarrow \Delta, P$. Clearly $\vdash \Pi(\mathcal{D}), P, \Gamma, \neg_*\Delta, \neg_* P \Rightarrow *$.

Case L$\bot$. Then $\mathcal{D}\colon \bot, \Gamma \Rightarrow \Delta$ with Γ, Δ atomic. If $(\bot, \Gamma) \cap \Delta = \emptyset$ then $\Pi(\mathcal{D}) = \{\bot \to *\}$ and hence $\vdash \Pi(\mathcal{D}), \bot, \Gamma, \neg_*\Delta \Rightarrow *$. If $(\bot, \Gamma) \cap \Delta \neq \emptyset$ then clearly $\vdash \Pi(\mathcal{D}), \bot, \Gamma, \neg_*\Delta \Rightarrow *$.

Case L$\to$. Then $\mathcal{D}$ ends with

$$\dfrac{\begin{array}{c} \mid \mathcal{D}_1 \\ \Gamma \Rightarrow \Delta, A \end{array} \qquad \begin{array}{c} \mid \mathcal{D}_2 \\ B, \Gamma \Rightarrow \Delta \end{array}}{A \to B, \Gamma \Rightarrow \Delta}\ \mathrm{L}{\to}$$

We have $\vdash \Pi(\mathcal{D}_1), \Gamma, \neg_*\Delta, \neg_* A \Rightarrow *$ and $\vdash \Pi(\mathcal{D}_2), B, \Gamma, \neg_*\Delta \Rightarrow *$ by induction hypothesis. Since $\Pi(\mathcal{D}) = \Pi(\mathcal{D}_1) \cup \Pi(\mathcal{D}_2)$, from the former we have $\vdash \Pi(\mathcal{D}), \Gamma, \neg_*\Delta \Rightarrow \neg_*\neg_* A$ and from the latter $\vdash \Pi(\mathcal{D}), \neg_*\neg_* B, \Gamma, \neg_*\Delta \Rightarrow *$. Hence $\vdash \Pi(\mathcal{D}), \neg_*\neg_* A \to \neg_*\neg_* B, \Gamma, \neg_*\Delta \Rightarrow *$ by L$\to$. But $\vdash (A \to B) \to \neg_*\neg_* A \to \neg_*\neg_* B$. Therefore $\vdash \Pi(\mathcal{D}), A \to B, \Gamma, \neg_*\Delta \Rightarrow *$.

Case R$\to$. Then $\mathcal{D}$ ends with

$$\dfrac{\begin{array}{c} \mid \mathcal{D}_1 \\ A, \Gamma \Rightarrow \Delta, B \end{array}}{\Gamma \Rightarrow \Delta, A \to B}\ \mathrm{R}{\to}$$

By induction hypothesis we have $\vdash \Pi(\mathcal{D}_1), A, \Gamma, \neg_*\Delta, \neg_* B \Rightarrow *$ and hence

$$\vdash \Pi(\mathcal{D}_1), \Gamma, \neg_*\Delta \Rightarrow A \to \neg_*\neg_* B.$$

Now $\mathrm{DNS}_{\to}$ gives

$$\vdash \mathrm{Peirce}_{*,B}, \Pi(\mathcal{D}_1), \Gamma, \neg_*\Delta \Rightarrow \neg_*\neg_*(A \to B).$$

Using the fact that Peirce suffices for the final atom we obtain

$$\vdash \Pi(\mathcal{D}), \Gamma, \neg_*\Delta, \neg_*(A \to B) \Rightarrow *,$$

since $\Pi(\mathcal{D}) = \Pi(\mathcal{D}_1) \cup \{\mathrm{Peirce}_{*,P}\}$ for P the final conclusion of B. $\square$

Corollary 3.5. *Let* $\mathcal{D}\colon \Gamma \Rightarrow A$ *in* **G3cp**. *Then* $\vdash \Pi, \Gamma \Rightarrow A$ *for some set* Π *of instances* $\mathrm{Peirce}_{Q,P}$ *of Peirce formulas with* Q *the final atom of* A, *plus possibly (depending on whether or not* $\bot$ *appears in* Γ, A*) the formula* $\bot \to Q$.

Proof. Let $A = A_1 \to \cdots \to A_n \to Q$. Since $A_1, \ldots A_n$ can be moved into Γ, it suffices to prove the claim with Q for A. By Proposition 3.4 from $\mathcal{D}\colon \Gamma \Rightarrow Q$ in **G3cp** we have $\vdash \Pi(\mathcal{D}), \Gamma, \neg_* Q \Rightarrow *$. Substituting Q for $*$ gives $\vdash \Pi, \Gamma \Rightarrow Q$ with $\Pi := \Pi(\mathcal{D})[* := Q]$. Because of the normalization theorem for **G3cp** we have the subformula property. Therefore for Γ, Q without $\bot$ a normal derivation $\mathcal{D}\colon \Gamma \Rightarrow Q$ in **G3cp** cannot involve $\bot$ altogether. Hence in this case Π consists of Peirce formulas only. $\square$

In the next section we will see that in general an unbounded number of Peirce formulas is necessary.

4. Examples

All implicational formulas[3] below do not contain $\bot$, and are provable in classical but not in minimal logic. In each case we provide a list of

(i) stability for the final atom and instances of ex-falso-quodlibet, and
(ii) instances of Peirce formulas

from which one can prove the example formula in minimal logic. In accordance with Proposition 3.4 we use $*$ for the final atom.

There is a general method to obtain such implicational formulas from well-known classical tautologies in the form of a disjunction: rewrite $A \vee B$ into $(A \to *) \to (B \to *) \to *$. We give some examples below.

4.1. Generalized Peirce formulas.

$$((* \to A_0) \to *) \to *,$$
$$((((* \to A_0) \to *) \to A_1) \to *) \to *,$$
$$((\dots((((* \to A_0) \to *) \to A_1) \to *) \dots \to A_k) \to *) \to *$$

can be derived from (i)

$$((* \to \bot) \to \bot) \to * \quad \text{and} \quad \bot \to A_0 \quad \dots \text{and} \quad \bot \to A_k$$

where all ex-falso-quodlibet formulas are necessary, and also (ii) from

$$\mathrm{Peirce}_{*,A_0},$$
$$\mathrm{Peirce}_{*,A_0} \quad \text{and} \quad \mathrm{Peirce}_{*,A_1},$$
$$\mathrm{Peirce}_{*,A_0} \quad \text{and} \quad \mathrm{Peirce}_{*,A_1} \quad \dots \text{and} \quad \mathrm{Peirce}_{*,A_k}.$$

To see that all Peirce formulas are necessary, suppose $\vdash (\mathrm{Peirce}_{*,A_j})_{j \neq i} \to \mathrm{GP}_n$, where GP_n is the n-th generalized Peirce formula. Replace all A_j ($j \neq i$) by $*$. Then the result GP'_n is equivalent to Peirce_{*,A_i} and hence $\vdash \mathrm{Peirce}_{*,A_i}$, a contradiction.

4.2. Nagata formulas.

This is another generalization of Peirce formulas.

$$N_{k+1}(*, A_0, \dots, A_k) := ((* \to N_k(A_0, \dots, A_k)) \to *) \to *.$$

with $N_0(A) := A$. Hence in particular

$$\begin{aligned} N_1(*, A) &= ((* \to A) \to *) \to * = \mathrm{Peirce}_{*,A}, \\ N_2(*, A, B) &= ((* \to N_1(A, B)) \to *) \to * \\ &= ((* \to ((A \to B) \to A) \to A) \to *) \to *. \end{aligned}$$

$N_{k+1}(*, A_0, \dots, A_k)$ can be derived from (i)

$$((* \to \bot) \to \bot) \to * \quad \text{and} \quad \bot \to A_0$$

and also (ii) from Peirce_{*,A_0}.

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[3]We are grateful to Pierluigi Minari for providing many of these examples.

4.3. **Minari formula.**

$$((* \to A) \to B) \to (B \to *) \to *$$

can be derived from (i)

$$((* \to \bot) \to \bot) \to * \quad \text{and} \quad \bot \to A$$

and also (ii) from $\text{Peirce}_{*,A}$.

4.4. **Mints formula.**

$$((((A \to B) \to A) \to A) \to *) \to *$$

can be derived from (i)

$$((* \to \bot) \to \bot) \to * \quad \text{and} \quad \bot \to B$$

and also (ii) from $\text{Peirce}_{*,B}$.

4.5. **Glivenko formula.**

$$(((B \to A) \to ((B \to C) \to A) \to A) \to *) \to *$$

can be derived from (i)

$$((* \to \bot) \to \bot) \to * \quad \text{and} \quad \bot \to C$$

and also (ii) from $\text{Peirce}_{*,C}$.

4.6. **Examples derived from classical disjunctive tautologies.** From the well-known classical tautologies

$$((A \to B) \to B) \vee (A \to B), \tag{3}$$

$$(B \to A) \vee (((A \to B) \to A) \to A), \tag{4}$$

$$A \vee (A \to B). \tag{5}$$

we obtain – by rewriting them as described above – non-trivial implicational example formulas for our general study. From (3) we obtain

$$(((A \to B) \to B) \to *) \to ((A \to B) \to *) \to *,$$

which can be derived from (i)

$$((* \to \bot) \to \bot) \to * \quad \text{and} \quad \bot \to B$$

and also (ii) from $\text{Peirce}_{*,B}$. From (4) we have

$$((B \to A) \to *) \to ((((A \to B) \to A) \to A) \to *) \to *,$$

which can be derived from (i)

$$((* \to \bot) \to \bot) \to * \quad \text{and} \quad \bot \to A$$

and also (ii) from $\text{Peirce}_{*,A}$. Finally (5) gives us

$$(A \to *) \to ((A \to B) \to *) \to *,$$

which can be derived from (i)

$$((* \to \bot) \to \bot) \to * \quad \text{and} \quad \bot \to B$$

and also (ii) from $\text{Peirce}_{*,B}$.

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