Termination of permutative conversions in intuitionistic Gentzen calculi

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Abstract

It is shown that permutative conversions terminate for the cut-free intuitionistic Gentzen (i.e. sequent) calculus; this proves a conjecture by Dyckhoff and Pinto. The main technical tool is a term notation for derivations in Gentzen calculi. These terms may be seen as λ-terms with explicit substitution, where the latter corresponds to the left introduction rules. *Keywords*: Termination of permutative conversions, explicit substitution, sequent term, natural deduction.

Prawitz [7] defines a map F transforming derivations in the intuitionistic Gentzen (or sequent) calculus **LJ** into natural deductions in **NJ**. Moreover he (essentially) proved surjectivity of F by constructing an inverse map G from **NJ** back to **LJ**. Howard [3] identified a class of cut-free sequent derivations – to be called π-normal below – corresponding to normal derivation terms (this is stated without proof in [3, Theorem 2]) [1]. A number of authors made use of 'permutative conversions' to clarify this situation. Zucker [9] showed for the negative fragment of $\mathbf{LJ}^c$ (i.e. **LJ** with cut) that two derivations have the same value under F iff they can be transformed into each other by means of permutative conversions (more precisely: permutative conversions in the sense of Kleene [4] and in addition permutations with the cut rule). Pottinger [6] extended Zucker's results to formulas with $\vee$ and $\exists$. Mints [5] and independently Dyckhoff and Pinto [1] prove a similar result for cut free calculi, where in Dyckhoff and Pinto [1] the orientation of permutative conversions is taken into account and confluence is proven. Here we prove termination of some versions of the permutative conversion rules; a weaker form of one of our results was conjectured in [1].

For simplicity we restrict attention to the negative fragment of intuitionistic logic (i.e. of minimal logic, since no symbol $\perp$ for falsity is present); however, the arguments below can be extended to the full language with $\vee, \exists$ (thanks to Matthias Hölzl who has checked this).

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1 Derivations as sequent terms

Cut-free derivations in the negative fragment of **LJ** are denoted by *sequent terms*; the difference to λ-terms and hence to derivation terms in natural deduction is that a form of *explicit substitution* is allowed. An inductive definition of sequent terms is displayed in table 1. Type (i.e. formula) indices will be omitted whenever possible. We use L, M, N, K for sequent terms and u, v, w for assumption variables.

Any sequent term L has a *type*, which is viewed as usual as the derived formula. The *context* of a sequent term L is the set $\Gamma := \mathrm{FA}(L)$ of free (typed) assumption variables of L, defined by

$$\begin{aligned}
\mathrm{FA}(u) &:= \{u\}, \\
\mathrm{FA}(\langle L, M\rangle) &:= \mathrm{FA}(L) \cup \mathrm{FA}(M), \\
\mathrm{FA}(L_u\{w, i\}) &:= (\mathrm{FA}(L) \setminus \{u\}) \cup \{w\},
\end{aligned}$$

[1] More recently, Herbelin [2] introduced still another sequent calculus LJT corresponding one-to-one with normal derivation terms (cf. also Troelstra [8, Proposition 2.2]). Herbelin proves termination of cut-elimination for LJT.

$$
\begin{aligned}
\mathrm{FA}(\lambda u L) &:= \mathrm{FA}(L) \setminus \{u\}, \\
\mathrm{FA}(M_u\{v, L\}) &:= (\mathrm{FA}(M) \setminus \{u\}) \cup \{v\} \cup \mathrm{FA}(L), \\
\mathrm{FA}(\lambda x L) &:= \mathrm{FA}(L), \\
\mathrm{FA}(L_u\{v, t\}) &:= (\mathrm{FA}(L) \setminus \{u\}) \cup \{v\}.
\end{aligned}
$$

The variable condition of course is $x \notin \mathrm{FV}(A)$ for all $u^A \in \Gamma := \mathrm{FA}(L)$.

So a context Γ is a set $\{u_1\colon A_1, \dots, u_n\colon A_n\}$ with distinct u_i. Sequent terms may be viewed as trees, where each node is labelled with a sequent of the form $\Gamma \Rightarrow A$, with $\Gamma := \mathrm{FA}(L)$ the context and A the type of L. The notation $\Gamma[u\colon A]$ in rule $\to$R u means that the context may or may not contain the assumption $u\colon A$.

It can happen that a sequent is derivable in more than one way. For instance, C can be derived from $u_1\colon A, u_2\colon A \to B, u_3\colon A \to B \to C$ by $u_u\{w, v\}_w\{u_3, u_1\}_v\{u_2, u_1\}$, i.e. by

$$
\dfrac{u_1\colon A \Rightarrow A \qquad \dfrac{u_1\colon A \Rightarrow A \qquad \dfrac{\dfrac{v\colon B \Rightarrow B \qquad u\colon C \Rightarrow C}{w\colon B \to C, v\colon B \Rightarrow C}}{}}{u_3\colon A \to B \to C, u_1\colon A, v\colon B \Rightarrow C}}{u_2\colon A \to B, u_1\colon A, u_3\colon A \to B \to C \Rightarrow C}
$$

and also by $u_u\{w, v_v\{u_2, u_1\}\}_w\{u_3, u_1\}$, i.e.

$$
\dfrac{u_1\colon A \Rightarrow A \qquad \dfrac{\dfrac{u_1\colon A \Rightarrow A \qquad v\colon B \Rightarrow B}{u_2\colon A \to B, u_1\colon A \Rightarrow B} \qquad u\colon C \Rightarrow C}{w\colon B \to C, u_2\colon A \to B, u_1\colon A \Rightarrow C}}{u_3\colon A \to B \to C, u_1\colon A, u_2\colon A \to B \Rightarrow C}
$$

2 Multiary normal forms

It will be useful to allow a form of multiary application for sequent terms. E.g. $M_u\{v, L_1 L_2\}$ should be essentially the same derivation as $M_u\{w, L_2\}_w\{v, L_1\}$ with a new assumption variable w. More generally we want to allow a multiary application of $\to$L and denote it by the sequent term $M_u\{v, L_1 \dots L_n\}$; this corresponds to the derivation

$$
\dfrac{\begin{matrix} \mid L_1 \\ \Gamma_1 \Rightarrow A_1 \end{matrix} \quad \dots \quad \begin{matrix} \mid L_n \\ \Gamma_n \Rightarrow A_n \end{matrix} \qquad \begin{matrix} \mid M \\ [u\colon B]\Delta \Rightarrow C \end{matrix}}{v\colon A_1 \to \dots A_n \to B, \Gamma_1, \dots, \Gamma_n, \Delta \Rightarrow C} \to\mathrm{L}^*\ u
$$

We also want to allow a multiary application of the rules $\to$L and $\wedge$L as in $M_u\{v, L_1 1 L_2\}$, i.e. in

$$
\dfrac{\begin{matrix} \mid L_1 \\ \Gamma_1 \Rightarrow A_1 \end{matrix} \qquad \begin{matrix} \mid L_2 \\ \Gamma_2 \Rightarrow A_2 \end{matrix} \qquad \begin{matrix} \mid M \\ [u\colon B]\Delta \Rightarrow C \end{matrix}}{v\colon A_1 \to D \wedge (A_2 \to B), \Gamma_1, \Gamma_2, \Delta \Rightarrow C} \wedge\!\to\!\mathrm{L}^*
$$

Also $\forall$L can be included: the sequent term $M_u\{v, L_1 t L_2 1 s\}$ corresponds to

$$
\dfrac{\begin{matrix} \mid L_1 \\ \Gamma_1 \Rightarrow A_1 \end{matrix} \qquad \begin{matrix} \mid L_2 \\ \Gamma_2 \Rightarrow (A_2)_x[t] \end{matrix} \qquad \begin{matrix} \mid M \\ [u\colon B_{x,y}[t,s]]\Delta \Rightarrow C \end{matrix}}{v\colon A_1 \to \forall x(A_2 \to D \wedge \forall y B), \Gamma_1, \Gamma_2, \Delta \Rightarrow C} \wedge\!\to\!\forall\mathrm{L}^*
$$

It should be clear now what is meant by a *multiary sequent term*. More precisely, simultaneously with the inductive definition of multiary sequent terms we have to define the notion of a *path* $\vec{L}$ for a formula A leading to (the instance) B of A; $\vec{L}$ will be a list consisting of multiary sequent terms, $0, 1$ (for projections) and object terms. The clauses are

- For every A, the empty list is a path for A leading to the instance A of A.
- If L^A is a multiary sequent term and $\vec{L}$ is a path for B leading to the instance C of B, then $L^A \vec{L}$ is a path for $A \to B$ leading to the instance C of $A \to B$.

Derivation	Term
$u\colon A \Rightarrow A$	u^A
$\dfrac{\begin{matrix} \mid L & & \mid M \\ \Gamma \Rightarrow A & & \Delta \Rightarrow B \end{matrix}}{\Gamma\Delta \Rightarrow A \wedge B} \wedge\mathrm{R}$	$\dfrac{L^A \qquad M^B}{\langle L^A, M^B \rangle^{A \wedge B}}$
$\dfrac{\begin{matrix} \mid L \\ [u\colon A]\Gamma \Rightarrow C \end{matrix}}{w\colon A \wedge B, \Gamma \Rightarrow C} \wedge\mathrm{L}_0\ u, w \qquad \dfrac{\begin{matrix} \mid L \\ [v\colon B]\Gamma \Rightarrow C \end{matrix}}{w\colon A \wedge B, \Gamma \Rightarrow C} \wedge\mathrm{L}_1\ v, w$	$\dfrac{L^C}{L^C_{u^A}\{w^{A \wedge B}, 0\}} \qquad \dfrac{L^C}{L^C_{v^B}\{w^{A \wedge B}, 1\}}$
$\dfrac{\begin{matrix} \mid L \\ \Gamma[u\colon A] \Rightarrow B \end{matrix}}{\Gamma \Rightarrow A \to B} \to\mathrm{R}\ u$	$\dfrac{L^B}{(\lambda u^A L^B)^{A \to B}}$
$\dfrac{\begin{matrix} \mid L & & \mid M \\ \Gamma \Rightarrow A & & [u\colon B]\Delta \Rightarrow C \end{matrix}}{v\colon A \to B, \Gamma, \Delta \Rightarrow C} \to\mathrm{L}\ u, v$	$\dfrac{L^A \qquad M^C}{M^C_{u^B}\{v^{A \to B}, L^A\}}$
$\dfrac{\begin{matrix} \mid L \\ \Gamma \Rightarrow A \end{matrix}}{\Gamma \Rightarrow \forall x A} \forall\mathrm{R}\ x$ if $x \notin \mathrm{FV}(\Gamma)$	$\dfrac{L^A}{(\lambda x L^A)^{\forall x A}}$ (with var. cond.)
$\dfrac{\begin{matrix} \mid L \\ [u\colon A_x[t]]\Gamma \Rightarrow B \end{matrix}}{v\colon \forall x A, \Gamma \Rightarrow B} \forall\mathrm{L}\ u, v, t$	$\dfrac{L^B}{L^B_{u^{A_x[t]}}\{v^{\forall x A}, t\}}$

Table 1: Gentzen calculus

- If t is an object term and $\vec{L}$ is a path for $A_x[t]$ leading to the instance B of $A_x[t]$, then $t\vec{L}$ is a path for $\forall x A$ leading to the instance B of $\forall x A$.
- If $\vec{L}$ is a path for A_i leading to the instance B of A_i, then $i\vec{L}$ is a path for $A_0 \wedge A_1$ leading to the instance B of $A_0 \wedge A_1$.

For example, $L_1^{A_1} t L_2^{(A_2)_x[t]} 1 s$ is a path for $A := A_1 \to \forall x(A_2 \to D \wedge \forall y B)$ leading to the instance $B_{x,y}[t,s]$ of A. – Then in the inductive definition of multiary sequent terms we have a single L-clause

- If $\vec{L}$ is a non empty path for A leading to the instance B of A, and M is a multiary sequent term of type C, then
$$M_{u^B}\{v^A, \vec{L}\}$$
is a multiary sequent term of type C.

We now want to explain precisely how a multiary sequent term can be viewed as an abbreviation of an ordinary (or 'binary') sequent term.

2.1. Definition. For multiary sequent terms we define an *abbreviation conversion* μ (μ for 'multiary') by

$$N_w\{u,\vec{M}\}_u\{v,\vec{L}\} \mapsto_\mu N_w\{v,\vec{L}\vec{M}\} \quad \text{if } u \notin \mathrm{FA}(N,\vec{M}).$$

The *term closure* $\longrightarrow_\mu$ of this conversion is defined inductively by

- If $M \mapsto_\mu M'$, then $M \longrightarrow_\mu M'$ (*head conversion*).
- If $M \longrightarrow_\mu M'$, then also

$$\begin{array}{rcl} M_u\{v,\vec{L}\} & \longrightarrow_\mu & M'_u\{v,\vec{L}\}, \\ L_u\{v,\vec{M}M\vec{N}\} & \longrightarrow_\mu & L_u\{v,\vec{M}M'\vec{N}\}, \\ \langle M,N\rangle & \longrightarrow_\mu & \langle M',N\rangle, \\ \langle N,M\rangle & \longrightarrow_\mu & \langle N,M'\rangle, \\ \lambda zM & \longrightarrow_\mu & \lambda zM' \quad \text{with } z\in\{u,x\}. \end{array}$$

These reductions are called *inner reductions*.

For example, the sequent term $u_u\{w,v\}_w\{u_3,u_1\}_v\{u_2,u_1\}$ considered above can be reduced in one $\longrightarrow_\mu$-step to $u_u\{u_3,u_1v\}_v\{u_2,u_1\}$, corresponding to

$$\dfrac{u_1\colon A\Rightarrow A \qquad \dfrac{u_1\colon A\Rightarrow A \qquad v\colon B\Rightarrow B \qquad u\colon C\Rightarrow C}{u_3\colon A\to B\to C, u_1\colon A, v\colon B\Rightarrow C}}{u_2\colon A\to B, u_1\colon A, u_3\colon A\to B\to C\Rightarrow C}$$

Similarly the second sequent term $u_u\{w,v_v\{u_2,u_1\}\}_w\{u_3,u_1\}$ can be reduced in one $\longrightarrow_\mu$-step to $u_u\{u_3,u_1v_v\{u_2,u_1\}\}$:

$$\dfrac{u_1\colon A\Rightarrow A \qquad \dfrac{u_1\colon A\Rightarrow A \qquad v\colon B\Rightarrow B}{u_2\colon A\to B, u_1\colon A\Rightarrow B} \qquad u\colon C\Rightarrow C}{u_3\colon A\to B\to C, u_1\colon A, u_2\colon A\to B\Rightarrow C}$$

Clearly $\longrightarrow_\mu$ is terminating, since the number of braces goes down. Local confluence of $\longrightarrow_\mu$ is also easy to see; in the overlap case we have

$$\begin{tikzcd}[column sep=tiny] & N_w\{u,\vec{M}\}_u\{v,\vec{L}\}_v\{u_1,\vec{K}\} \arrow[dl] \arrow[dr] & \\ N_w\{u,\vec{M}\}_u\{u_1,\vec{K}\vec{L}\} \arrow[dr] & & N_w\{v,\vec{L}\vec{M}\}_v\{u_1,\vec{K}\} \arrow[dl] \\ & N_w\{u_1,\vec{K}\vec{L}\vec{M}\} & \end{tikzcd}$$

Hence every multiary sequent term has a uniquely determined *μ-normal form* to be called *multiary normal form*. Sequent terms in multiary normal form are characterized by

$$L ::= u \mid \langle L,M\rangle \mid \lambda zL \mid M_u\{v,\vec{L}\},$$

with $z\in\{u,x\}$, where in the last case M is not of the form $N_w\{u,\vec{M}\}$ with $u\notin\mathrm{FA}(N,\vec{M})$ (for then $N_w\{u,\vec{M}\}_u\{v,\vec{L}\}$ could be further abbreviated by $N_w\{v,\vec{L}\vec{M}\}$). Recall that the list $\vec{L}$ may contain $0,1$ (for projections) as well as object terms. Also in case λxL the variable condition $x\notin\mathrm{FV}(A)$ for all $u^A\in\mathrm{FA}(L)$ must hold.

For instance, $u_u\{u_3,u_1v\}_v\{u_2,u_1\}$ is the multiary normal form of $u_u\{w,v\}_w\{u_3,u_1\}_v\{u_2,u_1\}$, and $u_u\{u_3,u_1v_v\{u_2,u_1\}\}$ is the multiary normal form of $u_u\{w,v_v\{u_2,u_1\}\}_w\{u_3,u_1\}$.

The multiary normal form $\mu(L)$ of a sequent term L can also be defined recursively. Moreover we define a kind of converse ν of μ, turning a multiary sequent term into a binary one.

2.2. Definition. 1. For every multiary sequent term L we define $\mu(L)$ by

$$\begin{array}{rcl}
\mu(u) & := & u, \\
\mu(\langle L, M\rangle) & := & \langle \mu(L), \mu(M)\rangle, \\
\mu(\lambda z L) & := & \lambda z \mu(L) \quad \text{with } z \in \{u, x\}, \\
\mu(M_u\{v, \vec{L}\}) & := & \begin{cases} N_w\{v, \mu(\vec{L})\vec{M}\} & \text{if } \mu(M) = N_w\{u, \vec{M}\} \text{ with } u \notin \mathrm{FA}(N, \vec{M}), \\ \mu(M)_u\{v, \mu(\vec{L})\} & \text{otherwise.} \end{cases}
\end{array}$$

2. For every multiary sequent term N we define a binary sequent term $\nu(N)$ by

$$\begin{array}{rcl}
\nu(u) & := & u, \\
\nu(\langle M, N\rangle) & := & \langle \nu(M), \nu(N)\rangle, \\
\nu(\lambda z N) & := & \lambda z \nu(N) \quad \text{with } z \in \{u, x\}, \\
\nu(N_w\{v, L\vec{M}\}) & := & \begin{cases} \nu(N)_w\{v, \nu(L)\} & & \text{if } \vec{M} \text{ is empty,} \\ \nu(N_w\{u, \vec{M}\})_u\{v, \nu(L)\} & \text{with } u \text{ new} & \text{otherwise.} \end{cases}
\end{array}$$

Clearly $\mu(L)$ is the uniquely determined multiary normal form of L.

2.3. Lemma. *For every binary sequent term L we have $\nu(\mu(L)) = L$.*

Proof. Induction on L. We only consider the case $M_u\{v, L\}$ with $\mu(M) = N_w\{u, \vec{M}\}$. If $u \notin \mathrm{FA}(N, \vec{M})$, then

$$\begin{array}{rcll}
\nu(\mu(M_u\{v, L\})) & = & \nu(N_w\{v, \mu(L)\vec{M}\}) & \\
& = & \nu(N_w\{u, \vec{M}\})_u\{v, \nu(\mu(L))\} & \text{wlog with } u, \text{ since } u \notin \mathrm{FA}(N, \vec{M}) \\
& = & \nu(\mu(M))_u\{v, \nu(\mu(L))\} & \\
& = & M_u\{v, L\} & \text{by ind. hyp.}
\end{array}$$

In case $u \in \mathrm{FA}(N, \vec{M})$ we have

$$\nu(\mu(M_u\{v, L\})) = \nu(\mu(M)_u\{v, \mu(L)\}) = \nu(\mu(M))_u\{v, \nu(\mu(L))\} = M_u\{v, L\}.$$ Qed

In what follows we shall simply speak of sequent terms when we mean multiary sequent terms.

Now recall the usual inductive definition of derivation terms, as displayed in table 2.3. The (actual) substitution of a derivation term N^A for an assumption variable u^A in the derivation term M^B is denoted by $M_u[N]$. Recall also that normal derivation terms (w.r.t. β-conversion $(\lambda x M)N \mapsto M_x[N]$) are characterized by

$$M ::= u\vec{M} \mid \langle M, N\rangle \mid \lambda z M$$

with $z \in \{u, x\}$, where in case $\lambda x M$ the obvious variable condition must hold.

The well known equivalence of the cut-free Gentzen calculus with normal derivations in natural deduction can be written quite explicitly with the present notation; this will be useful below.

2.4. Definition. 1. For every sequent term L we define a derivation term $F(L)$ (F for 'functional') by

$$\begin{array}{rcl}
F(u) & := & u, \\
F(\langle L, M\rangle) & := & \langle F(L), F(M)\rangle, \\
F(\lambda z L) & := & \lambda z F(L) \quad \text{with } z \in \{u, x\}, \\
F(M_u\{v, \vec{L}\}) & := & F(M)_u[vF(\vec{L})].
\end{array}$$

Derivation	Term
$u: A$	u^A
$\dfrac{\begin{matrix} \mid M \\ A \end{matrix} \qquad \begin{matrix} \mid M \\ B \end{matrix}}{A \wedge B} \wedge^+$	$\dfrac{M^A \qquad N^B}{\langle M^A, N^B \rangle^{A \wedge B}}$
$\dfrac{\begin{matrix} \mid M \\ A \wedge B \end{matrix}}{A} \wedge_0^- \qquad \dfrac{\begin{matrix} \mid M \\ A \wedge B \end{matrix}}{B} \wedge_1^-$	$\dfrac{M^{A \wedge B}}{(M0)^A} \qquad \dfrac{M^{A \wedge B}}{(M1)^B}$
$\dfrac{\begin{matrix} \mid M \\ B \end{matrix}}{A \to B} \to^+ u^A$	$\dfrac{M^B}{(\lambda u^A M)^{A \to B}}$
$\dfrac{\begin{matrix} \mid M \\ A \to B \end{matrix} \qquad \begin{matrix} \mid N \\ A \end{matrix}}{B} \to^-$	$\dfrac{M^{A \to B} \qquad N^A}{(MN)^B}$
$\dfrac{\begin{matrix} \mid M \\ A \end{matrix}}{\forall x A} \forall^+$ if $x \notin \mathrm{FV}(B)$ for u^B free in M	$\dfrac{M^A}{(\lambda x M)^{\forall x A}}$ (with var. cond.)
$\dfrac{\begin{matrix} \mid M \\ \forall x A \end{matrix} \qquad t}{A_x[t]} \forall^-$	$\dfrac{M^{\forall x A}}{(Mt)^{A_x[t]}}$

Table 2: Natural deduction

2. For every normal derivation term N we define a sequent term $G(N)$ (G for Gentzen) by

$$\begin{aligned} G(u) &:= u, \\ G(\langle M, N \rangle) &:= \langle G(M), G(N) \rangle, \\ G(\lambda z N) &:= \lambda z G(N) \quad \text{with } z \in \{u, x\}, \\ G(v\vec{N}) &:= u_u\{v, G(\vec{N})\} \quad \text{with } u \text{ new.} \end{aligned}$$

Here e.g. $F(\vec{L})$ abbreviates the list obtained by applying F to the elements of the list $\vec{L}$.

2.5. Lemma.

1. *For every sequent term L we have that $F(L)$ is a normal derivation term, deriving the same formula from no more assumptions.*
2. *For every normal derivation term N we have that $G(N)$ is a sequent term in multiary normal form, deriving the same formula from the same assumptions.*

3. For normal derivation terms N we have $F(G(N)) = N$.

PROOF 1. Induction on L. 2. Induction on N. 3. Induction on N. QED

So in this sense natural deduction and the Gentzen calculus are equivalent. In particular we have assigned to any sequent term L in a natural way a normal derivation term $F(L)$, essentially by executing the explicit substitutions (in the last clause of the definition of F).

2.6. REMARK. $v_n\colon A_n, w_n\colon A_n \to A_n \to A_{n-1}, \ldots, w_2\colon A_2 \to A_2 \to A_1, w_1\colon A_1 \to A_1 \to A \Rightarrow A$ can be derived by $L_n := u_u\{w_1, v_1 v_1\}_{v_1}\{w_2, v_2 v_2\}\cdots_{v_n}\{w_n, v_n v_n\}$, a sequent term of size linear in n. However, the normal derivation term $F(L_n) = u_u[w_1 v_1 v_1]_{v_1}[w_2 v_2 v_2]\cdots_{v_n}[w_n v_n v_n]$ has 2^n occurrences of v_n.

3 Permutative conversions

In the two example derivations considered above one can see that the second can be obtained from the first by a certain permutation of rules. The right premise of the lower application of $\to$L in

$$\dfrac{u_1\colon A \Rightarrow A \qquad \dfrac{u_1\colon A \Rightarrow A \qquad v\colon B \Rightarrow B \qquad u\colon C \Rightarrow C}{u_3\colon A \to B \to C, u_1\colon A, v\colon B \Rightarrow C}\to\mathrm{L}^*}{u_2\colon A \to B, u_1\colon A, u_3\colon A \to B \to C \Rightarrow C}\to\mathrm{L}$$

is an $\to$L-application itself. If one moves the lower $\to$L-application upwards to the place where the assumption variable to be cancelled appears – i.e. to the middle premise –, then we obtain the second derivation

$$\dfrac{u_1\colon A \Rightarrow A \qquad \dfrac{u_1\colon A \Rightarrow A \qquad v\colon B \Rightarrow B}{u_2\colon A \to B, u_1\colon A \Rightarrow B}\to\mathrm{L} \qquad u\colon C \Rightarrow C}{u_3\colon A \to B \to C, u_1\colon A, u_2\colon A \to B \Rightarrow C}\to\mathrm{L}^*$$

Here the right premise of any of the two $\to$L-rules is an axiom.

More generally by so-called *permutative conversions* we can achieve that in all 'left' or elimination rules $M_u\{w, i\}$, $M_u\{v, L\}$ and $M_u\{v, t\}$ the main premise M solely consists of the assumption variable u bound in this rule. We will see that these so-called π-*normal* sequent terms correspond bijectively to normal derivation terms.

To formulate permutative conversions we make use of the multiary normal form introduced above.

3.1. DEFINITION. 1. The relation $\mapsto_\pi$ of *permutative conversion* is defined by

$$\begin{array}{rcll} M_v\{u, \vec{L}\} & \mapsto_\pi & M, \quad \text{if } v \notin \mathrm{FA}(M). & (1)\\ N_w\{u_1, \vec{M}\}_v\{u, \vec{L}\} & \mapsto_\pi & N_w\{u_1, \mu(\vec{M}_v\{u, \vec{L}\})\}, \quad \text{if } u_1 \neq v,\ v \notin \mathrm{FA}(N) \text{ and } v \in \mathrm{FA}(\vec{M}). & (2)\\ N_w\{v, \vec{M}\}_v\{u, \vec{L}\} & \mapsto_\pi & N_w\{u, \vec{L}\mu(\vec{M}_v\{u, \vec{L}\})\}, \quad \text{if } v \notin \mathrm{FA}(N) \text{ and } v \in \mathrm{FA}(\vec{M}). & (3)\\ \langle M, N\rangle_v\{u, \vec{L}\} & \mapsto_\pi & \langle \mu(M_v\{u, \vec{L}\}), \mu(N_v\{u, \vec{L}\})\rangle. & (4)\\ (\lambda z M)_v\{u, \vec{L}\} & \mapsto_\pi & \lambda z.\mu(M_v\{u, \vec{L}\}) \quad \text{for } z \in \{w, x\} \text{ and } w \neq v. & (5) \end{array}$$

Observe that from $M \mapsto_\pi M'$ with M in multiary normal form we can conclude that M' is in multiary normal form too. – The *term closure* $\longrightarrow_\pi$ of these conversions is defined as in definition 2.1

2. For any sequent terms M, M' let $M \longrightarrow_\pi M'$ iff for the multiary normal forms $\mu(M)$ and $\mu(M')$ we have $\mu(M) \longrightarrow_\pi \mu(M')$ as defined in 1.

We now prove termination of $\longrightarrow_\pi$; clearly it suffices to consider sequent terms in multiary normal form. Our main tool will be a measure function δ, defined as follows.

3.2. DEFINITION.

$$\begin{array}{rcl} \delta(u) & := & 1,\\ \delta(\langle L, M\rangle) & := & \delta(L) + \delta(M),\\ \delta(\lambda z L) & := & \delta(L) \quad \text{with } z \in \{u, x\},\\ \delta(M_u\{v, \vec{L}\}) & := & \delta(M) + \#(M)\cdot\left(1 + \sum \delta(\vec{L})\right). \end{array}$$

Here we have made use of $\#(M)$ (a bound for the number of nodes in M), which is defined by

$$\begin{aligned}
\#(u) &:= 1, \\
\#(\langle L, M\rangle) &:= \#(L) + \#(M) + 1, \\
\#(\lambda z L) &:= \#(L) + 1 \quad \text{with } z \in \{u, x\}, \\
\#(M_u\{v, \vec{L}\}) &:= \#(M) \cdot \left(1 + \sum \#(\vec{L})\right).
\end{aligned}$$

We first show that the introduction of abbreviations in the sense of $\longrightarrow_\mu$ also lowers the δ-measure.

3.3. Lemma. *If $M \longrightarrow_\mu M'$, then $\delta(M) > \delta(M')$ and $\#(M) > \#(M')$.*

Proof. It suffices to prove the claim for the case $M \mapsto_\mu M'$; the general case then follows from the definition of δ. So we consider a conversion $N_w\{u, \vec{M}\}_u\{v, \vec{L}\} \mapsto_\mu N_w\{v, \vec{L}\vec{M}\}$ with $u \notin \mathrm{FA}(N, \vec{M})$. For the left hand side we have

$$\begin{aligned}
\delta(N_w\{u, \vec{M}\}_u\{v, \vec{L}\}) &= \delta(N_w\{u, \vec{M}\}) + \#(N_w\{u, \vec{M}\}) \left(1 + \sum \delta(\vec{L})\right) \\
&= \delta(N) + \#(N) \left(1 + \sum \delta(\vec{M})\right) + \#(N) \left(1 + \sum \#(\vec{M})\right) \left(1 + \sum \delta(\vec{L})\right)
\end{aligned}$$

and for the right hand side

$$\delta(N_w\{v, \vec{L}\vec{M}\}) = \delta(N) + \#(N) \left(1 + \sum \delta(\vec{L}) + \sum \delta(\vec{M})\right).$$

Clearly the lhs is greater than the rhs. Similarly we have for $\#$ on the lhs

$$\#(N_w\{u, \vec{M}\}_u\{v, \vec{L}\}) = \#(N) \left(1 + \sum \#(\vec{M})\right) \left(1 + \sum \#(\vec{L})\right)$$

and on the rhs

$$\#(N_w\{v, \vec{L}\vec{M}\}) = \#(N) \left(1 + \sum \#(\vec{L}) + \sum \#(\vec{M})\right).$$

Again the lhs clearly is greater than the rhs. Qed

3.4. Theorem. *If $M \longrightarrow_\pi M'$, then $\delta(M) > \delta(M')$ and $\#(M) \geq \#(M')$.*

Proof. We first treat the permutative conversions. By Lemma 3.3 it suffices to do this for the terms without μ on the right hand side.

(1). Let $v \notin \mathrm{FA}(M)$. Then we have

$$\delta(M_v\{u, \vec{L}\}) = \delta(M) + \#(M) \left(1 + \sum \delta(\vec{L})\right) > \delta(M),$$

since $\#(M) > 0$, and

$$\#(M_v\{u, \vec{L}\}) = \#(M) \left(1 + \sum \#(\vec{L})\right) \geq \#(M).$$

(2). Let $u_1 \neq v$, $v \notin \mathrm{FA}(N)$ and $v \in \mathrm{FA}(\vec{M})$. Then we have for the lhs

$$\begin{aligned}
\delta(N_w\{u_1, \vec{M}\}_v\{u, \vec{L}\}) &= \delta(N_w\{u_1, \vec{M}\}) + \#(N_w\{u_1, \vec{M}\}) \left(1 + \sum \delta(\vec{L})\right) \\
&= \delta(N) + \#(N) \left(1 + \sum \delta(\vec{M})\right) + \#(N) \left(1 + \sum \#(\vec{M})\right) \left(1 + \sum \delta(\vec{L})\right)
\end{aligned}$$

and for the rhs

$$\begin{aligned}
\delta(N_w\{u_1, \vec{M}_v\{u, \vec{L}\}\}) &= \delta(N) + \#(N) \left(1 + \sum \delta(\vec{M}_v\{u, \vec{L}\})\right) \\
&= \delta(N) + \#(N) \left(1 + \sum \delta(\vec{M}) + \left(\sum \#(\vec{M})\right) \left(1 + \sum \delta(\vec{L})\right)\right).
\end{aligned}$$

Clearly the lhs is greater than the rhs. For # we have on the lhs

$$\begin{aligned}
\#(N_w\{u_1,\vec{M}\}_v\{u,\vec{L}\}) &= \#(N_w\{u_1,\vec{M}\})\left(1+\sum\#(\vec{L})\right)\\
&= \#(N)\left[1+\sum\#(\vec{M})\right]\left(1+\sum\#(\vec{L})\right)
\end{aligned}$$

and for the rhs

$$\begin{aligned}
\#(N_w\{u_1,\vec{M}_v\{u,\vec{L}\}\}) &= \#(N)\left(1+\sum\#(\vec{M}_v\{u,\vec{L}\})\right)\\
&= \#(N)\left[1+\left(\sum\#(\vec{M})\right)\left(1+\sum\#(\vec{L})\right)\right].
\end{aligned}$$

Clearly the lhs is greater than or equal to the rhs.

(3). Let $v\notin \mathrm{FA}(N)$ and $v\in\mathrm{FA}(\vec{M})$. Then we have for the lhs

$$\begin{aligned}
\delta(N_w\{v,\vec{M}\}_v\{u,\vec{L}\}) &= \delta(N_w\{v,\vec{M}\})+\#(N_w\{v,\vec{M}\})\left(1+\sum\delta(\vec{L})\right)\\
&= \delta(N)+\#(N)\left(1+\sum\delta(\vec{M})\right)+\#(N)\left(1+\sum\#(\vec{M})\right)\left(1+\sum\delta(\vec{L})\right).
\end{aligned}$$

For the rhs we obtain

$$\begin{aligned}
&\delta(N_w\{u,\vec{L}\vec{M}_v\{u,\vec{L}\}\})\\
&= \delta(N)+\#(N)\left(1+\sum\delta(\vec{L})+\sum\delta(\vec{M}_v\{u,\vec{L}\})\right)\\
&= \delta(N)+\#(N)\left(1+\sum\delta(\vec{M})\right)+\#(N)\sum\delta(\vec{L})+\#(N)\left(\sum\#(\vec{M})\right)\left(1+\sum\delta(\vec{L})\right).
\end{aligned}$$

Clearly again the lhs is greater than the rhs. For # we have on the lhs

$$\begin{aligned}
\#(N_{u_1}\{v,\vec{M}\}_v\{u,\vec{L}\}) &= \#(N_{u_1}\{v,\vec{M}\})\left(1+\sum\#(\vec{L})\right)\\
&= \#(N)\left(1+\sum\#(\vec{M})\right)\left(1+\sum\#(\vec{L})\right)
\end{aligned}$$

and for the rhs

$$\begin{aligned}
\#(N_{u_1}\{u,\vec{L}\vec{M}_v\{u,\vec{L}\}\}) &= \#(N)\left(1+\sum\#(\vec{L})+\sum\#(\vec{M}_v\{u,\vec{L}\})\right)\\
&= \#(N)\left(1+\sum\#(\vec{L})+\left(\sum\#(\vec{M})\right)\left(1+\sum\#(\vec{L})\right)\right).
\end{aligned}$$

So in this case the lhs equals the rhs.

Among the cases (4) and (5) we only treat (5) with $w\neq v$. We obtain for the lhs

$$\delta((\lambda wM)_v\{u,\vec{L}\})=\delta(\lambda wM)+\#(\lambda wM)\left(1+\sum\delta(\vec{L})\right)=\delta(M)+(\#(M)+1)\left(1+\sum\delta(\vec{L})\right).$$

For the rhs we have

$$\delta(\lambda w.M_v\{u,\vec{L}\})=\delta(M_v\{u,\vec{L}\})=\delta(M)+\#(M)\left(1+\sum\delta(\vec{L})\right).$$

So again the lhs is greater than the rhs. Concerning # we have for the lhs

$$\#((\lambda wM)_v\{u,\vec{L}\})=\#(\lambda wM)\left(1+\sum\#(\vec{L})\right)=(\#(M)+1)\left(1+\sum\#(\vec{L})\right)$$

and for the rhs

$$\#(\lambda w.M_v\{u,\vec{L}\})=\#(M_v\{u,\vec{L}\})+1=\#(M)\left(1+\sum\#(\vec{L})\right)+1.$$

Again the lhs is greater than or equal to the rhs.

For the inner reductions the claim follows immediately from the definitions of δ and #. QED

3.5. LEMMA. *A sequent term in multiary normal form is* π-normal *iff subterms of the form* $M_v\{u,\vec{L}\}$ *only occur with* $M = v$.

PROOF. The direction from right to left is clear. For the converse let a sequent term in multiary normal form be given and assume that it contains a subterm of the form $M_v\{u,\vec{L}\}$ with $M \neq v$, where we may assume that M has the required form. Then M is not an assumption variable because of rule (1) of definition 3.1, not a pair because of (4) and not an abstraction because of (5). So M is of the form $w_w\{u_1,\vec{M}\}$ and hence $M_v\{u,\vec{L}\}$ of the form

$$w_w\{u_1,\vec{M}\}_v\{u,\vec{L}\}.$$

Case $u_1 \neq v$. Reducibility follows from (1) if $v \notin \mathrm{FA}(\vec{M})$, and from (2) if $v \in \mathrm{FA}(\vec{M})$. *Case* $u_1 = v$. Then we have $v \in \mathrm{FA}(\vec{M})$, since we only consider terms in multiary normal form. Now reducibility follows from (3). QED

3.6. LEMMA.

1. *For normal derivation terms* N *we have* $F(G(N)) = N$.
2. *For* π*-normal sequent terms* L *we have* $G(F(L)) = L$.
3. *If* $L \longrightarrow_\pi L'$, *then* $F(L) = F(L')$.

PROOF 1. This is Lemma 2.5(3). 2. Induction on L, using Lemma 3.5; for instance we have

$$G(F(u_u\{v,\vec{L}\})) = G(vF(\vec{L})) = u_u\{v,\vec{L}\}.$$

3. Induction on the definition of $L \longrightarrow_\pi L'$. The permutative conversions are defined in such a way that execution of the explicit substitutions produces identities. QED

3.7. COROLLARY. *(Dyckhoff and Pinto [1]).*

1. *(Uniqueness of normal form). Any sequent term* L *has the uniquely determined* π*-normal form* $G(F(L))$.
2. *(Characterization of* $\sim_\pi$*). Let* $\sim_\pi := \stackrel{*}{\longleftrightarrow}_\pi$, *i.e. the equivalence relation generated by* $\longrightarrow_\pi$. *Then we have* $L_1 \sim_\pi L_2$ *iff* $F(L_1) = F(L_2)$.

PROOF 1. By Theorem 3.4 we can construct a π-normal form L^* of L. Then we have $F(L) = F(L^*)$ by Lemma 3.6(3), hence $G(F(L)) = G(F(L^*)) = L^*$ by Lemma 3.6(2).

2. From $L_1 \sim_\pi L_2$ we obtain $F(L_1) = F(L_2)$ by Lemma 3.6(3). The converse follows from $L_i \stackrel{*}{\longrightarrow}_\pi G(F(L_i))$, which holds by part 1. QED

Finally we prove a conjecture by Dyckhoff and Pinto [1] concerning termination of a certain version of the permutative conversion rules (2) and (3) considered there. In these rules they require (instead of our restriction $v \notin \mathrm{FA}(N)$) that N has the form $(w_1)_{w_1}\{w,\vec{N}\}$ with $w \notin \mathrm{FA}(\vec{N})$, where now $v \in \mathrm{FA}(\vec{N})$ is allowed. So we now have instead of (2) and (3) the rules

$$\begin{aligned}
(w_1)_{w_1}\{w,\vec{N}\}_w\{u_1,\vec{M}\}_v\{u,\vec{L}\} &\mapsto_\pi (w_1)_{w_1}\{w,\vec{N}_v\{u,\vec{L}\}\}_w\{u_1,\vec{M}_v\{u,\vec{L}\}\}, \\
&\qquad \text{if } u_1 \neq v,\ w \notin \mathrm{FA}(\vec{N}) \text{ and } v \in \mathrm{FA}(\vec{N},\vec{M}). \qquad (6)\\
(w_1)_{w_1}\{w,\vec{N}\}_w\{v,\vec{M}\}_v\{u,\vec{L}\} &\mapsto_\pi (w_1)_{w_1}\{w,\vec{N}_v\{u,\vec{L}\}\}_w\{v_1,\vec{M}_v\{u,\vec{L}\}\}_{v_1}\{u,\vec{L}\}, \\
&\qquad \text{if } w \notin \mathrm{FA}(\vec{N}) \text{ and } v \in \mathrm{FA}(\vec{N},\vec{M}). \qquad (7)
\end{aligned}$$

To see termination we form the multiary normal forms on both sides.

$$\begin{aligned}
(w_1)_{w_1}\{u_1,\mu(\vec{M}\vec{N})\}_v\{u,\mu(\vec{L})\} &\longrightarrow_\pi (w_1)_{w_1}\{u_1,\mu(\vec{M}_v\{u,\vec{L}\}\vec{N}_v\{u,\vec{L}\})\}, \\
&\qquad \text{if } u_1 \neq v \text{ and } v \in \mathrm{FA}(\vec{N},\vec{M}). \\
(w_1)_{w_1}\{v,\mu(\vec{M}\vec{N})\}_v\{u,\mu(\vec{L})\} &\longrightarrow_\pi (w_1)_{w_1}\{u,\mu(\vec{L}\vec{M}_v\{u,\vec{L}\}\vec{N}_v\{u,\vec{L}\})\}, \quad \text{if } v \in \mathrm{FA}(\vec{N},\vec{M}).
\end{aligned}$$

Each time we reach the right hand side from the left hand side by (2) or (3). Hence, as shown above, the δ-values decrease.

More precisely, Dyckhoff and Pinto consider a slightly different form of (6) and (7), where instead of $(w_1)_{w_1}\{w, \vec{N}_v\{u, \vec{L}\}\}$ they still have $(w_1)_{w_1}\{w, \vec{N}\}_v\{u, \vec{L}\}$ (which because of $v \neq w, w_1$ could be reduced to the former). To see that also this original form of their conversion terminates we introduce yet another simplification relation $\longrightarrow_\sigma$, to be called σ-reduction and defined as the term closure of $(w_1)_{w_1}\{w, \vec{N}\}_v\{u, \vec{L}\} \mapsto_\sigma (w_1)_{w_1}\{w, \vec{N}_v\{u, \vec{L}\}\}$ for $v \neq w$. It is easy to see that $\longrightarrow_\sigma$ terminates (the number of occurrences of pairs of braces of the form $\ldots\}\ldots\{\ldots$ decreases), that the σ-normal form is unique (easy) and that the δ-values decrease under $\longrightarrow_\sigma$ (easy calculation). Then in any reduction step the δ-values of the σ-normal forms decrease, and hence we have termination. – Dyckhoff and Pinto suggest that π-normality of $\vec{N}, \vec{M}, \vec{L}$ would be sufficient; however, this is far too strong and is not necessary for termination.

References

[1] Roy Dyckhoff and Luis Pinto. Permutability of proofs in intuitionistic sequent calculi. This volume.

[2] Hugo Herbelin. A λ-calculus structure isomorphic to Gentzen-style sequent calculus structure. In L. Pacholski and J. Tiuryn, editors, *Computer Science Logic. 8th Workshop, CSL'94. Kazimierz, Poland, September 1994*, volume 933 of *LNCS*, pages 61–75. Springer Verlag, Berlin, Heidelberg, New York, 1995.

[3] W.A. Howard. The formulae–as–types notion of construction. In J.P. Seldin and J.R. Hindley, editors, *To H.B. Curry: Essays on Combinatory Logic, Lambda Calculus and Formalism*, pages 479–490. Academic Press, 1980.

[4] Stephen C. Kleene. Permutability of inferences in Gentzen's calculi LK and LJ. Memoirs of the American Mathematical Society No. 10, pp. 1–26, Providence, 1952.

[5] Gregory Mints. Normal forms for sequent derivations. In P. Odifreddi, editor, *Kreiseliana. About and Around Georg Kreisel*, pages 469–492. A.K. Peters, Wellesley, Massachusetts, 1996.

[6] G. Pottinger. Normalization as a homomorphic image of cut–elimination. *Annals of Mathematical Logic*, 12:323–357, 1977.

[7] Dag Prawitz. *Natural Deduction*, volume 3 of *Acta Universitatis Stockholmiensis. Stockholm Studies in Philosophy*. Almqvist & Wiksell, Stockholm, 1965.

[8] Anne S. Troelstra. Marginalia on sequent calculi. To appear in Studia Logica.

[9] J. I. Zucker. The correspondence between cut–elimination and normalization I, II. *Annals of Mathematical Logic*, 7:1–156, 1974.