Mathematisches Institut
der Universität München
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Wintersemester 2006/7
Blatt 6

Übungen zur Vorlesung "Recursion Theory"

Aufgabe 21. Show that the following relations are arithmetical and estimate their position in the arithmetical hierarchy.

(a) $\leq_f$ is a linear ordering of $\mathbb{N}$ and n is the $\leq_f$-least element of $\mathbb{N}$;
(b) $\leq_f$ is a discrete linear ordering of $\mathbb{N}$ (i.e., every element of $\mathbb{N}$ has an immediate $\leq_f$-successor);
(c) $\leq_f$ is a well-ordering $\mathbb{N}$ with order type ω;
(d) $\leq_f$ is a well-ordering $\mathbb{N}$ with order type $\leq \omega \cdot n$ (a relation in f and n).

Aufgabe 22. (a) Show that the set $\mathcal{D}_1$ of all unary functions whose graph is in Σ^0_1 is arithmetical, and estimate its position in the arithmetical hierarchy.
(b) Do the same for the set $\mathcal{D}_r$ of all unary functions whose graph is in Δ^0_r.

Aufgabe 23. Let $P(\vec{f}, \vec{g}, \vec{n})$ be an elementary relation of arity $(2r+1+p, q)$ and let the relation $Q(\vec{g}, \vec{n})$ of arity (p, q) be defined as

$$Q(\vec{g}, \vec{n}) \;\leftrightarrow\; \exists_{f_1} \forall_{f_2} \dots \exists_{f_{2r+1}} P(f_1, \dots, f_{2r+1}, \vec{g}, \vec{n}).$$

Prove that $P \in \Sigma^1_{2r}$. *Hint.* Recall that a $(1,1)$-ary relation R is Σ^0_1-definable if and only if it can be written in the form

$$R(g, n) \;\leftrightarrow\; \exists_s T_1(e, n, \overline{g}(s)).$$

Aufgabe 24. Let U be a fixed non-empty set. A map $\Gamma \colon \mathcal{P}(U) \to \mathcal{P}(U)$ is called an *operator* on U. Γ is called *monotone* if for all $X, Y \subseteq U$ from $X \subseteq Y$ we can conclude $\Gamma(X) \subseteq \Gamma(Y)$.

$$\mathcal{I}_\Gamma := \bigcap \{\, X \subseteq U \mid \Gamma(X) \subseteq X \,\}$$

is the set defined inductively by the monotone operator Γ. Prove that

$$\mathcal{I}_\Gamma = \mathcal{I}'_\Gamma := \bigcap \{\, X \subseteq U \mid \Gamma(X) = X \,\}.$$

Abgabetermin. Donnerstag, 30. November 2006, 11:15 Uhr