Homework for the lecture course „Mathematical Logic“

Problem 45. (4 points). Let T be a theory as in Section 3.2 of the course.

(a) Prove: if $M, N \subseteq \mathbb{N}$ are representable in T, then $M \cap N$ is as well.
(b) Let $f, g\colon \mathbb{N} \to \mathbb{N}$ be representable in T by formulas $A(x, y)$ and $B(y, z)$. Prove that then $g \circ f$ is representable in T as well.

Problem 46. (4 points). Let T be a theory as in Section 3.2 of the course. A closed formula G is called Gödel sentence if

$$T \vdash G \leftrightarrow \neg P(\underline{\ulcorner G \urcorner}) \qquad \text{where } P(x) := \exists_y B_{\mathrm{Prf}_T}(y, x).$$

(a) Prove that a Gödel sentence exists.
(b) Let G be a Gödel sentence. Assume that (i) for all closed formulas A we have: if $T \vdash A$, then also $T \vdash P(\underline{\ulcorner A \urcorner})$, and (ii) $T \nvdash \bot$. Prove that $T \nvdash G$.

Problem 47. (4 points). Let NatGcd be defined by

$$\begin{aligned}
\mathrm{NatGcd}(0, n) &:= n\\
\mathrm{NatGcd}(n, 0) &:= n\\
\mathrm{NatGcd}(Sn, Sm) &:= \begin{cases} \mathrm{NatGcd}(S(n), m - n) & \text{if } n < m\\ \mathrm{NatGcd}(n - m, S(m)) & \text{otherwise} \end{cases}
\end{aligned}$$

Prove in Minlog (s. ueb12.scm)

(a) NatGcdSelf: $\forall_n(\mathrm{NatGcd}(n, n) = n)$
(b) NatGcdOne: $\forall_n(\mathrm{NatGcd}(1, n) = 1)$
(c) NatGcdComm: $\forall_{n,m}(\mathrm{NatGcd}(n, m) = \mathrm{NatGcd}(m, n))$

Problem 48. (4 points). Let $\mathrm{NatDiv}(n, m) := \exists_{l \leq m}(l \cdot n = m)$. Prove in Minlog (s. ueb12.scm)

(a) NatGcdDiv0: $\forall_{n,m}(\mathrm{NatDiv}(\mathrm{NatGcd}(n, m), n))$
(b) NatGcdDiv1: $\forall_{n,m}(\mathrm{NatDiv}(\mathrm{NatGcd}(n, m), m))$
(c) NatDivGcd:
$\forall_{n,m,l}(\mathrm{NatDiv}(l, n) \to \mathrm{NatDiv}(l, m) \to \mathrm{NatDiv}(l, \mathrm{NatGcd}(n, m)))$

Due. Wednesday, 21. January 2026, 8:00.