Homework for the lecture course „Mathematical Logic“

Problem 41. (4 points). Let $\mathcal{L}$ be the language given by the function symbols 0 (nullary) uand S (unary) and the relation symbol $=$ (binary). For $k \in \mathbb{N}$ define the numeral $\underline{k} \in \mathrm{Ter}_{\mathcal{L}}$ by $\underline{0} := 0$ and $\underline{k+1} := S\underline{k}$. Define an elementary function s such that for every $\mathcal{L}$-formula $C = C(z)$ with $z := x_0$ we have

$$s(\ulcorner C \urcorner, k) = \ulcorner C(\underline{k}) \urcorner.$$

It suffices to define such a function s. You may use that

$$s_{\mathrm{Ter}}(n,k) := \begin{cases} \ulcorner \underline{k} \urcorner & \text{if } n = \langle \mathrm{sn}(\mathrm{Var}), 0 \rangle, \\ n & \text{if } n = \langle \mathrm{sn}(\mathrm{Var}), (n)_1 \rangle \text{ and } 0<(n)_1, \\ n & \text{if } n = \langle \mathrm{sn}(0) \rangle, \\ \langle \mathrm{sn}(S), s_{\mathrm{Ter}}((n)_1, k) \rangle & \text{if } n = \langle \mathrm{sn}(S), (n)_1 \rangle, \\ n & \text{else} \end{cases}$$

for all $\mathcal{L}$-terms $t = t(z)$ with $z := x_0$ has the property $s_{\mathrm{Ter}}(\ulcorner t \urcorner, k) = \ulcorner t(\underline{k}) \urcorner$ and is elementary.

Problem 42. (4 points). Let T be a theory as in Section 3.1.2 of the course, $A(x_1, \dots, x_n, y)$ a formula and $f \colon \mathbb{N}^n \to \mathbb{N}$ a function. Assume

(1) $T \vdash A(\underline{a_1}, \dots, \underline{a_n}, \underline{f(a_1, \dots, a_n)})$ for all $a_1, \dots, a_n \in \mathbb{N}$,

(2) $T \vdash A(\underline{a_1}, \dots, \underline{a_n}, y) \to A(\underline{a_1}, \dots, \underline{a_n}, z) \to y{=}z$ for all $a_1, \dots, a_n \in \mathbb{N}$,

(3) $T \vdash \underline{b} \neq \underline{c}$ for $b < c$.

Prove that $T \vdash \neg A(\underline{a_1}, \dots, \underline{a_n}, \underline{c})$ if $c \neq f(a_1, \dots, a_n)$.

Problem 43. (4 points). Let $\mathrm{NatDiv}(n,m) := \exists_{l \le m}(l \cdot n = m)$. Prove in Minlog (cf. ueb11.scm)

(a) NatDivRefl: $\forall_n \mathrm{NatDiv}(n,n)$.
(b) NatDivTrans: $\forall_{l,m,n}(\mathrm{NatDiv}(l,m) \to \mathrm{NatDiv}(m,n) \to \mathrm{NatDiv}(l,n))$.
(c) NatDivZero: $\forall_n \mathrm{NatDiv}(n,0)$.
(d) NatDivToLe: $\forall_{n,m}(0 < m \to \mathrm{NatDiv}(n,m) \to n \le m)$.

Problem 44. (4 points). Prove in Minlog (cf. ueb11.scm)

(a) NatDivAntiSym: $\forall_{n,m}(\mathrm{NatDiv}(m,n) \to \mathrm{NatDiv}(n,m) \to n = m)$.
(b) NatDivPlus: $\forall_{l,m,n}(\mathrm{NatDiv}(l,m) \to \mathrm{NatDiv}(l,n) \to \mathrm{NatDiv}(l,m+n))$.
(c) NatDivTimes: $\forall_{l,m,n}(\mathrm{NatDiv}(l,m) \to \mathrm{NatDiv}(l,n \cdot m))$.
(d) NatDivPlusRev: $\forall_{l,m,n}(\mathrm{NatDiv}(l,m) {\to} \mathrm{NatDiv}(l,m+n) {\to} \mathrm{NatDiv}(l,n))$.

Due. Wednesday, 14. January 2026, 8:00.