Mathematisches Institut der LMU
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Winter semester 2025/26
hwk08

Homework for the lecture course „Mathematical Logic“

Problem 29. (4 points). The binary representation of positive natural numbers $\mathbb{P}$ is given by the constructors 1, S_0 (double) and S_1 (double +1). We define $\mathrm{NP}\colon \mathbb{N} \to \mathbb{P}$ and $\mathrm{PN}\colon \mathbb{P} \to \mathbb{N}$ by

$$\mathrm{NP}(0) = \mathrm{NP}(S(0)) = 1, \quad \mathrm{NP}(S(S(n))) = \begin{cases} S_0(\mathrm{NP}(S(H(n)))) & \text{if } E(n) \\ S_1(\mathrm{NP}(S(H(n)))) & \text{else} \end{cases}$$

$$\mathrm{PN}(1) = S(0), \quad \mathrm{PN}(S_0(p)) = D(\mathrm{PN}(p)), \quad \mathrm{PN}(S_1(p)) = S(D(\mathrm{PN}(p))).$$

Prove informally

(a) $\mathrm{NP}(\mathrm{PN}(p)) = p$,
(b) $\mathrm{PN}(\mathrm{NP}(S(n))) = S(n)$.

The functions H ($n \mapsto \lfloor n/2 \rfloor$), D ($n \mapsto 2n$) and E ($n \mapsto$ „n is even“) and their properties can be used without proof. Hint. (a). Induction on p. Without proof one may use:

$$\mathrm{NPDouble}: \quad 0 < n \to \mathrm{NP}(D(n)) = S_0(\mathrm{NP}(n)),$$
$$\mathrm{NPSuccDouble}: \quad 0 < n \to \mathrm{NP}(S(D(n))) = S_1(\mathrm{NP}(n)).$$

(b). Induction on n. Without proof one may use:

$$\mathrm{CVInd}: \quad \forall_n (\forall_{m<n} \mathit{ws}(m) \to \mathit{ws}(n)) \to \forall_n \mathit{ws}(n)$$

with ws a variable of type $\mathbb{N} \to \mathbb{B}$.

Problem 30. (4 points). Prove

(a) For every subelementary function $f\colon \mathbb{N}^r \to \mathbb{N}$ there is a natural number k such that for all $\vec{n} = n_1, \dots, n_r$ we have

$$f(\vec{n}\,) < 2^{k(\max(\vec{n},1))}.$$

(b) The function $n \mapsto 2^{n \cdot n}$ is not subelementary.

Problem 31. (4 points). Define the following programs by explicitly giving instructions, i.e., base instructions and the program constructs defined in the course (Section 2.1.2).

(a) $P(x; y)$ computes 2^x.
(b) $H(x; y)$ computes $\lfloor \frac{x}{2} \rfloor$, i.e., the largest number $\leq \frac{x}{2}$.
(c) $L(x; y)$ computes the integer logarithm for base 2, i.e., for $0 < x$ the number y with $2^y \leq x < 2^{y+1}$.

Problem 32. (4 points). Formalize the derivations from Problem 29 (see `ueb08.scm`).

Due. Wednesday, 10. Dezember 2025, 8:00.