Mathematisches Institut der LMU — Winter semester 2025/26

Helmut Schwichtenberg — hwk07

Homework for the lecture course „Mathematical Logic“

Problem 25. (4 points). Addition for natural numbers is defined by

$$n + 0 := n, \qquad n + Sm := S(n + m).$$

Prove informally that $+$ is commutative. Hint. Induction on m. Some auxiliary facts on $+$ are needed, which have to be proved as well.

Problem 26. (4 points). (Logic for decidable predicates). Let p, q be variables of the base type $\mathbb{B}$ of boolean objects, consisting of $\mathtt{tt}$ (true) and $\mathtt{ff}$ (false). Terms now have types, built from base types by $\to_{\mathrm{Typ}}$ (short: $\to$). The single predicate symbol is atom of arity $(\mathbb{B})$. Atomic formulas are $\mathrm{atom}(t)$ (short: t) with t a term of type $\mathbb{B}$. Apart from the constructors $\mathtt{tt}$, $\mathtt{ff}$ we consider function symbols (program constants), here only $=_{\mathbb{B}}$ (short: $=$) of type $\mathbb{B} \to \mathbb{B} \to \mathbb{B}$ with the computation rules

$$(\mathtt{tt} = \mathtt{tt}) := \mathtt{tt}, \qquad (\mathtt{tt} = \mathtt{ff}) := \mathtt{ff}, \qquad (\mathtt{ff} = \mathtt{tt}) := \mathtt{ff}, \qquad (\mathtt{ff} = \mathtt{ff}) := \mathtt{tt}.$$

Computation rules are to be read as replacement rules (from left to right). Two terms are called equal if they have a common reduct. Let $\mathbf{F} := (\mathtt{ff} = \mathtt{tt})$. Prove that $\mathbf{F} \to \forall_{p,q}(p = q)$ is derivable from axioms

$$\mathrm{CaseDist}\colon A(\mathtt{tt}) \to A(\mathtt{ff}) \to \forall_p A(p).$$

Problem 27. (4 points). Find programs for

$x = \min(y, z)$ writes the minimum of y, z in the register x,

$x = y \bmod 2$ write the remainder of division of y by 2 in x.

Use *exclusively* the base instructions

Zero: $x := 0$,

Succ: $x := x + 1$,

Jump: $[\textbf{if } x = y \textbf{ then } I_n \textbf{ else } I_m]$.

Problem 28. (4 points). Formalize the derivation from Problem 25 (see `ueb07.scm`).

Due. Wednesday, 3. Dezember 2025, 8:00.